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TWO CARDINAL VERSIONS OF DIAMOND

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ABSTRACT

Roughly speaking, $\diamondsuit_{\kappa,\lambda}$ asserts the existence of a sequence of size $< \kappa$ sets that captures every subset of λ on a stationary set. The paper is devoted to the study of $\diamondsuit_{\kappa,\lambda}$ and related principles, which are for instance obtained by considering sequences of larger sets, or by requesting the simultaneous capture of many subsets of λ. Our main result is that $\diamondsuit_{\kappa,\lambda}$ holds in case $\lambda > 2^{<\kappa}$.

0. Introduction

Let κ be a regular uncountable cardinal, and let S be a subset of κ. Jensen [10] introduced the following combinatorial principle: $\diamondsuit_\kappa(S)$ asserts the existence of a sequence s_α, $\alpha < \kappa$, that "captures" each subset A of κ on a stationary subset of S, by which we mean that $\{\alpha \in S\colon s_\alpha = A \cap \alpha\} \in NS_\kappa^+$. The starred version of diamond, $\diamondsuit_\kappa^*(S)$ states that there exist $w_\alpha \in [P(\alpha)]^{\leq|\alpha|}$, $\alpha < \kappa$, such that $\{\alpha \in S\colon A \cap \alpha \notin w_\alpha\} \in NS_\kappa$ for all $A \subseteq \kappa$. If S is stationary, then by a result of Kunen, $\diamondsuit_\kappa^*(S)$ implies $\diamondsuit_\kappa(S)$, as $\diamondsuit_\kappa(S)$ holds iff there are $w_\alpha \in [P(\alpha)]^{\leq|\alpha|}$, $\alpha < \kappa$, such that $\{\alpha \in S\colon A \cap \alpha \in w_\alpha\} \in NS_\kappa^+$ for all $A \subseteq \kappa$.

A two cardinal version of diamond was introduced by Jech in [9]. Let κ be a regular uncountable cardinal, let $\lambda \geq \kappa$ be a cardinal, and let $S \subseteq [\lambda]^{<\kappa}$. Then

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$\diamondsuit_{\kappa,\lambda}(S)$ asserts the existence of a sequence s_a, $a \in [\lambda]^{<\kappa}$, such that

$$\{a \in S: s_a = A \cap a\} \in NS^+_{\kappa,\lambda} \quad \text{for all } A \subseteq \lambda.$$

Such a sequence is called a $\diamondsuit_{\kappa,\lambda}(S)-$ sequence. To see that we are dealing here with a generalization of the one cardinal principle, observe that $\diamondsuit_{\kappa,\kappa}(S)$ is equivalent to $\diamondsuit_\kappa(S \cap \kappa)$, and that $\kappa \in NS^*_{\kappa,\kappa}$. Whereas the original principle has been widely used in applications, it has not been so for its analogue for two cardinals. $\diamondsuit_{\kappa,\lambda}(S)$ is however not trivial, as it implies that S can be split into $\lambda^{<\kappa}$ many pairwise disjoint stationary subsets. S has thus to be a "large" (i.e. of maximal cardinality) subset of $[\lambda]^{<\kappa}$. Now a stationary subset of $[\lambda]^{<\kappa}$ is not necessarily large, unless the GCH is assumed. Moreover, Gitik [7] showed that even if all stationary subsets of $[\lambda]^{<\kappa}$ are large, some of them may fail to have the splitting property. This is in contrast to a result of Solovay that states that every stationary subset of κ splits into κ many pairwise disjoint stationary subsets.

It has therefore seemed appropriate to us to prelude our study of diamond with a few sections dealing with the size of various stationary subsets of $[\lambda]^{<\kappa}$.

After setting some notation in Section 1, we consider in Section 2 some (in)-equalities that are relevant to the computation of $s(\kappa, \lambda)$, which is the least cardinality of any stationary set in $[\lambda]^{<\kappa}$. Except for a striking recent result of Shelah [21] , which is given without proof, all other results are by now folklore.

Section 3 opens with a characterization of the closed unbounded filter $NS^*_{\kappa,\lambda}$. Briefly, $D \in NS^*_{\kappa,\lambda}$ iff there is $F: \lambda \times \lambda \to \lambda$ such that D contains all $a \in [\lambda]^{<\kappa}$ such that $F[a \times a] \subseteq a$ and $a \cap \kappa \in \kappa$. The result goes back to Kueker [11], who used functions defined on $[\lambda]^{<\omega}$, and in the present formulation is due to Baumgartner (see [5]). It is natural to wonder whether one can do without the extra condition on $a \cap \kappa$. It is well-known that one can in case $\kappa = \omega_1$. The case $\kappa > \omega_1$, where the picture is no longer so clear-cut, has been treated by Feng [6]. We observe that the condition can be somewhat relaxed. For instance in case $\kappa = \omega_{n+1}$, it is enough to require that $a \cap \omega_n$ is unbounded in ω_n. It is known that assuming $\lambda > \kappa$, the equivalence above no longer holds, if a single variable function $G: \lambda \to \lambda$ is substituted for F. Still, one may ask whether it is possible to get away with functions $G: \mu \times \lambda \to \lambda$, where μ is some fixed cardinal $< \lambda$. Proposition 3.3 answers that question, except for the case when λ is the successor of a singular cardinal, which remains open.

The end of Section 3 and the whole of Section 4 are devoted to the study of some special stationary sets. We attempt to generalize some results of Baumgartner

[2]. Assume we are given two finite sets $A \subseteq [\kappa, \lambda]$ and $B \subseteq \kappa$ of regular cardinals, and $\varphi: A \to B$. Let T be the set of all $a \in [\lambda]^{<\kappa}$ such that

$$\bigcup(a \cap \mu) \in \{\alpha < \mu: \operatorname{cof}(\alpha) = \varphi(\mu)\} \quad \text{for all } \mu \in A.$$

Then T is stationary. Moreover, assuming there is some $\nu \in A$ such that $\nu > \kappa$ and $\varphi(\nu) = \omega$, there are at least $\nu^{\aleph_0}$ distinct members of $T \cap D$ for all $D \in NS^*_{\kappa,\lambda}$. It is easy to think of several ways to strengthen this statement. For one thing, we show that one can find $\nu^{\aleph_0}$ *incomparable* members of $T \cap D$ in case ν is finitely many cardinals away from κ. Another way to improve the statement would be to replace in the definition of T $\{\alpha < \mu: \operatorname{cof}(\alpha) = \varphi(\mu)\}$ by an arbitrary stationary subset of $\{\alpha < \mu: \operatorname{cof}(\alpha) < \kappa\}$. We show in Proposition 9.4 that it can be done in case $\kappa = \omega_1$.

Fix a regular cardinal $\mu < \kappa$, and consider the set of all $a \in \kappa$ with $\operatorname{cof}(a) = \mu$. Such an a is internally accessible, in the sense that it has a subset b of size μ with $\bigcup_{\beta \in b} \beta = a$. Section 5 is devoted to a brief study of a two cardinal version of this notion. Clearly if $\lambda > \kappa$, then the coding of a by b cannot in general be accomplished via the identity function $h(\beta) = \beta$; so one has instead recourse to a fixed $h: \lambda \to [\lambda]^{<\kappa}$. This approach is especially helpful when one discusses $\diamondsuit_{\kappa,\lambda}$ in situations where the GCH is not assumed. This has already been done by Shelah in [20].

Several attempts have been made to adapt the method used by Gregory [8] in his proof of $\diamondsuit^*_\kappa$ to the two cardinal situation. Section 6 is the account of one of them, actually a rather crude one, as it yields diamond, and not even diamond star.

We see in Section 7 how to obtain $\diamondsuit_{\kappa,\lambda}(S)$-sequences by forcing. Here and in Section 14 we only use simple forcing notions, namely those for adding Cohen subsets of a regular uncountable cardinal. As is well-known, forcing one Cohen subset of ν yields diamond at ν (and thus collapses $2^{<\nu}$ to ν in the process). It yields actually much more, i.e. $\diamondsuit_{\nu,\lambda}(S)$-sequences for all $\lambda > \nu$, as well as $\diamondsuit_{\kappa,\nu}(S)$-sequences for all $\kappa < \nu$. For the bottom-up direction, our main result is that adding one Cohen subset of ω_1 is enough to get $\diamondsuit_{\omega_1,\lambda}(S)$ for every $\lambda \geq \omega_1$ and every stationary S in the ground model. As regards the other, top-down, direction, we have the result of [15]: adding one Cohen subset of λ gives $\diamondsuit_{\kappa,\lambda}(S)$ for all $\kappa \leq \lambda$ and all stationary S in the ground model. As an aside, let us remark that this construction requires λ to be regular. We do not know how to handle the case when λ is singular (and of cofinality $\geq \kappa$, which seems to be the most interesting case). It is easy to extract from the proof a stronger principle, $\diamondsuit_{\kappa:\, \lambda:\, \lambda}(S)$, which reads as follows : there are $B_\alpha \subseteq \lambda$, $\alpha < \lambda$, such that $\{a \in S: B_{\cup a} \cap a = A \cap a\} \in NS^+_{\kappa,\lambda}$ for all $A \subseteq \lambda$. As $\diamondsuit_{\kappa:\, \kappa:\, \kappa}(S)$ simply is a reformulation of $\diamondsuit_{\kappa,\kappa}(S)$, one can see $\diamondsuit_{\kappa:\, \lambda:\, \lambda}(S)$ as an alternative generalization of the one cardinal diamond. The little we know about that new principle is expounded in Section 7.

Section 8 presents yet another two cardinal version of diamond. This time the idea is to modify the definition so as to make possible the capture of more than one set at a time. To be more accurate, we intend to capture families (as opposed to sequences) of sets. Is there an absolute upper limit on the size of the families that can be thus captured? That is of course to be expected, but we have been unable to show that any such limit exists. Let us now state our principle: $\diamondsuit^{\lambda^{<\kappa}}_{\kappa,\lambda}(S)$ asserts the existence of a sequence $t_a \subseteq P(a)$, $a \in [\lambda]^{<\kappa}$, such that $\{a \in S: t_a = \{A \cap a: A \in E\}\} \in NS^+_{\kappa,\lambda}$ for all $E \subseteq P(\lambda)$ with $|E| < \lambda^{<\kappa}$. Let us first observe that $\diamondsuit^{\kappa^{<\kappa}}_{\kappa,\kappa}(S)$ is easily shown to be equivalent to $\diamondsuit_{\kappa,\kappa}(S)$. As for the case when $\lambda > \kappa$, it seems that once $\diamondsuit_{\kappa,\lambda}(S)$ has been established, the truth-value of $\diamondsuit^{\lambda^{<\kappa}}_{\kappa,\lambda}(S)$ very much depends on the prevailing cardinal arithmetic. In any case, we show in Corollary 10.4 that assuming the GCH, $\diamondsuit_{\kappa,\lambda}(S)$ implies the apparently stronger $\diamondsuit^{\lambda^{<\kappa}}_{\kappa,\lambda}(S)$. Diamond principles have originated in the study of the constructible universe (where of course the GCH holds), and this may explain why $\diamondsuit^{\lambda^{<\kappa}}_{\kappa,\lambda}$ has not appeared in the literature before.

Shelah showed in [20] that $\diamondsuit_{\omega_1,\mu^+}(S)$ holds for some S provided $\mu^{\aleph_0} = \mu$. Section 9 is devoted to a generalization of that result. Let ν be a regular cardinal with $\kappa < \nu \leq \lambda$, and set $T = \{a \in [\lambda]^{<\kappa}: \mathrm{cof}(\bigcup(a \cap \nu)) = \omega\}$. As was remarked above, we have $|T| \geq \nu^{\aleph_0}$, as T is somewhat ramified. T is however not necessarily large. For example, if $\lambda = \kappa^+$, then $|T|$ equals $(\kappa^+)^{\aleph_0}$, which may be less than $(\kappa^+)^{<\kappa}$. We show that $\diamondsuit_{\kappa,\lambda}(T)$ holds if we assume that $2^{<\kappa} < \nu$. Notice that the assumption insures that every stationary subset of $[\lambda]^{<\kappa}$ is large. To get our result, we modify the games that Shelah used in his proof. Extra care in the definition of the games allows us to extend the result to the case when $\kappa = \omega_1$ and $\lambda = 2^{\aleph_0}$, assuming that $2^{\aleph_0}$ is regular, and that $2^{\aleph_0} > \omega_1$. The second assumption is clearly necessary, as Jensen showed that $\diamondsuit_{\omega_1}$ does not follow from CH, but the status of the first one is not so clear.

Let $\diamondsuit_{\kappa,\lambda,\lambda}(S)$ denote the following assertion : there are s_a, $a \in [\lambda]^{<\kappa}$, such that

$\{a \in S\colon s_a = A \cap \bigcup a\} \in NS^+_{\kappa,\lambda}$ for all $A \subseteq \lambda$. Here again, we have that $\diamondsuit_{\kappa,\kappa}(S)$ is (trivially) equivalent to $\diamondsuit_{\kappa,\kappa,\kappa}(S)$. The principle $\diamondsuit_{\kappa,\lambda,\lambda}(S)$ is the last one of the paper in our series of two cardinal versions of diamond. It is considered in Section 10, where we show it to be implied by (and thus equivalent to) $\diamondsuit_{\kappa,\lambda}(S)$ under the GCH.

Many more diamond sequences can be defined from a given diamond sequence. It is for example shown in Section 11 that assuming the GCH, $\diamondsuit_{\kappa,\lambda}(S)$ implies $\diamondsuit_{\kappa,\nu}(\{a\colon a \cap \lambda \in S\})$ for every cardinal $\nu > \lambda$. Provided certain conditions are fulfilled (see Proposition 11.4), it is also possible to go down from $\diamondsuit_{\kappa,\lambda}$ to $\diamondsuit_{\mu,\lambda}$, where $\mu < \kappa$. Here again we follow in the footsteps of Shelah [20].

The remainder of the paper is devoted to the study of starred versions of some of the principles considered above. $\diamondsuit^{*\lambda^{<\kappa}}_{\kappa,\lambda}(S)$ and $\diamondsuit^*_{\kappa,\lambda,\lambda}(S)$ are respectively featured in Section 12 and in Section 13. It is shown in Section 12 that assuming the GCH, $\diamondsuit^*_{\kappa,\lambda}(S)$ holds for every $S \subseteq \{a \in [\lambda]^{<\kappa}\colon \mathrm{cof}(\bigcup a) \neq \mathrm{cof}(|a|)\}$. One may wonder whether that result is sharp, i.e. whether there are models of the GCH where $\diamondsuit^*_{\kappa,\lambda}(S)$ fails for every stationary $S \subseteq \{a \in [\lambda]^{<\kappa}\colon \mathrm{cof}(\bigcup a) = \mathrm{cof}(|a|)\}$. The answer is immediate in case $\mathrm{cof}(\lambda) < \kappa$ and κ is the successor of a cardinal ν with $\mathrm{cof}(\nu) \neq \mathrm{cof}(\lambda)$. One also gets a positive answer when $\kappa = \lambda$, and when κ is the successor of a cardinal of cofinality ω and λ is regular. Those results and related forcing constructions can be found in Section 14.

The results of Sections 9, 10, 12 and 14 are joint work of the authors. Sections 2-8, 11 and 13 are due to the second author.

1. Notation

We let On denote the class of all ordinals. Given $\alpha, \beta \in \mathrm{On}$ with $\alpha \leq \beta$, we let $(\alpha,\beta) = \{\gamma \in \mathrm{On}\colon \alpha < \gamma < \beta\}, [\alpha,\beta) = (\alpha,\beta) \cup \{\alpha\}, (\alpha,\beta] = (\alpha,\beta) \cup \{\beta\}$ and $[\alpha,\beta] = (\alpha,\beta) \cup \{\alpha,\beta\}$. We set $\aleph^{+\beta}_\alpha = \aleph_{\alpha+\beta}$.

Given an ideal I over a set X, we put $I^+ = \{B \subseteq X\colon B \notin I\}$ and $I^* = \{B \subseteq X\colon X - B \in I\}$. Given a set X and a cardinal ν, we let

$$[X]^\nu = \{B \subseteq X\colon |B| = \nu\}, [X]^{<\nu} = \bigcup_{\nu' < \nu} [X]^{\nu'} \text{ and } [X]^{\leq \nu} = [X]^{<\nu} \cup [X]^\nu.$$

Throughout the paper κ will denote a regular uncountable cardinal, and λ a cardinal with $\lambda \geq \kappa$. Let X be a set of size $\geq \kappa$, and let $A \subseteq [X]^{<\kappa}$. A is said to be unbounded in case $[X]^{<\kappa} = \bigcup_{a \in A} P(a)$. A is closed if for every sequence

$a_\alpha \in A$, $\alpha < \gamma < \kappa$, such that $a_\beta \subseteq a_\alpha$ for $\beta < \alpha$, we have $\bigcup_{\alpha<\gamma} a_\alpha \in A$. By a result of Solovay, A is closed iff A is closed under directed unions of size $< \kappa$. A is stationary if $A \cap B \neq 0$ for every closed unbounded subset B of $[X]^{<\kappa}$. Nonstationary subsets of $[X]^{<\kappa}$ form an ideal, which we denote by $NS_{\kappa,X}$. We put $NS_\kappa = NS_{\kappa,\kappa} \cap P(\kappa)$.

We let $ND_{\kappa,\lambda}$ denote the set of all $S \subseteq [\lambda]^{<\kappa}$ such that $\diamondsuit_{\kappa,\lambda}(S)$ does not hold. We set $ND_\kappa = ND_{\kappa,\kappa} \cap P(\kappa)$. For any $S \subseteq [\lambda]^{<\kappa}$, $\diamondsuit^*_{\kappa,\lambda}(S)$ asserts the existence of a sequence $w_a \in [P(a)]^{\leq|a|}$, $a \in [\lambda]^{<\kappa}$, such that for all $A \subseteq \lambda$, $\{a \in S: A \cap a \notin w_a\} \in NS_{\kappa,\lambda}$. $\diamondsuit_{\kappa,\lambda}(S)$ follows from $\diamondsuit^*_{\kappa,\lambda}(S)$ in case $S \in NS^+_{\kappa,\lambda}$, as $\diamondsuit_{\kappa,\lambda}(S)$ holds whenever there are $w_a \in [P(a)]^{\leq|a|}$, $a \in [\lambda]^{<\kappa}$, such that for all $A \subseteq \lambda$, $\{a \in S: A \cap a \in w_a\} \in NS^+_{\kappa,\lambda}$. We let $D^*_{\kappa,\lambda}$ be the set of all $S \subseteq [\lambda]^{<\kappa}$ such that $\diamondsuit^*_{\kappa,\lambda}(S)$ holds.

For any set a, $\hat{a}$ will denote a fixed bijection from $|a|$ onto a. Given an infinite limit ordinal α, $\tilde{\alpha}$ will denote a strictly increasing function from $\operatorname{cof}(\alpha)$ onto some fixed closed unbounded subset C of α of order type $\operatorname{cof}(\alpha)$.

Given cardinals $\mu \geq \omega$ and $\nu > 0$, and a set I with $|I| \geq \mu$, $\mathrm{Fn}(I,\nu,\mu)$ denotes the set of all functions p such that $\mathrm{dom}(p) \in [I]^{<\mu}$ and $\mathrm{ran}(p) \subseteq \nu$, ordered by reverse inclusion. For each cardinal $\rho > 0$, $\mathrm{Fn}(\rho \times \mu, 2, \mu)$ is the poset for adding ρ many Cohen subsets of μ.

2. Unbounded subsets of $[\lambda]^{<\kappa}$

We let $s(\kappa,\lambda)$ (respectively $u(\kappa,\lambda)$) be the smallest cardinality of any stationary (resp. unbounded) subset of $[\lambda]^{<\kappa}$.

The second author asked whether $u(\kappa,\lambda)$ and $s(\kappa,\lambda)$ are equal. Shelah [21] recently showed that they are:

PROPOSITION 2.1: $s(\kappa,\lambda) = u(\kappa,\lambda)$.

PROPOSITION 2.2: (i) $u(\kappa,\lambda) \geq \lambda$.
(ii) $\operatorname{cof}(u(\kappa,\lambda)) \geq \kappa$.

Proof: (i) Use the fact that $\lambda = \cup E$ for every unbounded subset E of $[\lambda]^{<\kappa}$.

(ii) Suppose otherwise, and let E be an unbounded subset of $[\lambda]^{<\kappa}$ with $|E| = u(\kappa,\lambda)$. Pick $E_\alpha \in [E]^{<u(\kappa,\lambda)}$, $\alpha < \operatorname{cof}(u(\kappa,\lambda))$, so that $E = \bigcup_{\alpha<\operatorname{cof}(u(\kappa,\lambda))} E_\alpha$. For each α, choose $a_\alpha \in [\lambda]^{<\kappa}$ such that for every $e \in E_\alpha$, $a_\alpha - e \neq 0$. Then clearly $\bigcup_{\alpha<\operatorname{cof}(u(\kappa,\lambda))} a_\alpha - e \neq 0$ for all $e \in E$, a contradiction. ∎

PROPOSITION 2.3: $\lambda^{<\kappa} = 2^{<\kappa} \cdot u(\kappa, \lambda)$.

Proof: Simply notice that $[\lambda]^{<\kappa} = \bigcup_{e \in E} P(e)$ for every unbounded subset E of $[\lambda]^{<\kappa}$. ∎

Thus $u(\kappa, \lambda) = \lambda^{<\kappa}$ whenever $\lambda \geq 2^{<\kappa}$.

COROLLARY 2.4: *Assume λ is a strong limit with $\mathrm{cof}(\lambda) < \kappa$. Then $u(\kappa, \lambda) = 2^{\lambda}$.*

Proof: We have $2^{<\kappa} < \lambda$ and $\lambda^{<\kappa} = 2^{\lambda}$. ∎

PROPOSITION 2.5: *Let ν, μ, ρ be cardinals such that $\nu \in [\kappa, \rho], \mathrm{cof}(\nu) = \nu$, $\nu \leq \mu^+$, $\kappa \leq \mu$ and $\lambda \leq \rho$. Then $u(\kappa, \lambda) \leq u(\kappa, \mu) \cdot u(\nu, \rho)$.*

Proof: Let E and H be given such that E is unbounded in $[\rho]^{<\nu}$ and H is unbounded in $[\mu]^{<\kappa}$. We claim that the set of all $\hat{a}[d \cap |a|] \cap \lambda$, $a \in E$ and $d \in H$, is unbounded in $[\lambda]^{<\kappa}$. Given $b \in [\lambda]^{<\kappa}$, start by picking $a \in E$ with $b \subseteq a$, and then select $d \in H$ with $\hat{a}^{-1}[b] \subseteq d$. We have $b \subseteq \hat{a}[d \cap |a|] \cap \lambda$. ∎

PROPOSITION 2.6: *Assume λ is a limit with $\lambda > \kappa$.*

(i) *If $\mathrm{cof}(\lambda) \geq \kappa$, then $u(\kappa, \lambda) = \bigcup_{\kappa \leq \nu < \lambda} u(\kappa, \nu)$.*

(ii) *If $\mathrm{cof}(\lambda) < \kappa$, and if $\nu_\alpha \geq \kappa$, $\alpha < \mathrm{cof}(\lambda)$, is an increasing and cofinal in λ sequence of cardinals, then $u(\kappa, \lambda) \leq (\bigcup_{\alpha < \mathrm{cof}(\lambda)} u(\kappa, \nu_\alpha))^{\mathrm{cof}(\lambda)}$.*

Proof: (i) Set $\lambda = \bigcup_{\alpha < \mathrm{cof}(\lambda)} \nu_\alpha$, where each ν_α is a cardinal with $\kappa \leq \nu_\alpha < \lambda$. Now observe that if E_α is an unbounded subset of $[\nu_\alpha]^{<\kappa}$ for each α, then $\bigcup_{\alpha < \mathrm{cof}(\lambda)} E_\alpha$ is an unbounded subset of $[\lambda]^{<\kappa}$.

(ii) For each $\alpha < \mathrm{cof}(\lambda)$, let E_α be an unbounded subset of $[\nu_\alpha]^{<\kappa}$. Given $d \in [\lambda]^{<\kappa}$, pick $a_\alpha \in E_\alpha$, $\alpha < \mathrm{cof}(\lambda)$, with $d \cap \nu_\alpha \subseteq a_\alpha$. Then $d \subseteq \bigcup_{\alpha < \mathrm{cof}(\lambda)} a_\alpha$. ∎

3. Closed unbounded subsets of $[\lambda]^{<\kappa}$

PROPOSITION 3.1: *Let $D \in NS^*_{\kappa,\lambda}$. Then there exists F: $\{(\beta, \alpha) \in \lambda \times \lambda \colon \beta \leq \alpha\} \to \lambda$ such that $a \in D$ whenever a satisfies the following conditions :*

(i) *$a \in [\lambda]^{<\kappa} - \{0\}$;*

(ii) *$F(\beta, \alpha) \in a$ whenever $\beta, \alpha \in a$ with $\beta \leq \alpha$;*

(iii) *for every successor cardinal $\nu \in \bigcup(a \cap \kappa)$, there exists $n < \omega$ such that $\nu^{+n} \in \bigcup(a \cap \kappa)$ and $|a \cap \nu^{+n}| = \nu^{+n}$.*

Proof: By induction on the size of d, define $a_d \in D$, $d \in [\lambda]^{<\omega}$, so that $d \subseteq a_d$, and $a_c \subseteq a_d$ for $c \subseteq d$. Given $n \in [1, \omega)$, let f_{n+1}: $[\lambda]^{n+1} \to \lambda$ satisfy the following conditions. Suppose $d \in [\lambda]^n$, and let d_p, $p < n$, be the increasing enumeration of d. Then

(0) $f_{n+1}(d \cup \{d_{n-1}+1\}) = |a_d|$;

(1) $f_{n+1}(d \cup \{d_{n-1}+2k+2\}) = \widehat{a_d}(d_k)$ whenever $k < n$ and $d_k \in |a_d|$;

(2) $f_{n+1}(d \cup \{d_{n-1}+2k+3\}) = \widehat{a_{d-\{d_k\}}}(d_k)$ whenever $k < n$ and $d_k \in |a_{d-\{d_k\}}|$.

Also define g_i: $[\lambda]^2 \to \lambda$, $i < 2$, so that

(3) $g_0(\beta, \alpha) = \hat{\alpha}(\beta)$ whenever $\beta < |\alpha|$;

(4) $g_1(\beta, \alpha) = \hat{a}^{-1}(\beta)$.

Now fix a one-to-one function j: $[\lambda]^2 \to \lambda$. Given h: $[\lambda]^{n+1} \to \lambda$, where $n \in [1, \omega)$, define $J(h)$: $\lambda \to \lambda$ as follows. Choose h_q: $[\lambda]^q \to \lambda$, $q \in [1, n+1]$, so that

(5) $h_{n+1} = h$;

(6) $h_{p+1}(d_0, \ldots, d_p) = h_p(\{j(d_k, d_{k+1})\colon k < p\})$;

set $J(h) = h_1$.

Then define F: $\{(\beta, \alpha) \in \lambda \times \lambda\colon \beta \leq \alpha\} \to \lambda$ so that

(7) $F(\alpha, \alpha) = \alpha + 1$;

(8) $F(\alpha, \alpha + 1) = 0$;

(9) $F(\alpha, \alpha + 2) = J(g_i)(\gamma)$ whenever $\alpha = j(i, i+1+\gamma)$ and $i < 2$;

(10) $F(\alpha, \alpha + 2) = J(f_{n+1})(\gamma)$ whenever $\alpha = j(n+1, n+2+\gamma)$ and $n \in [1, \omega)$;

(11) $F(\alpha, \delta + 2) = j(\alpha, \delta)$ whenever $\alpha < \delta$.

Let $a \in [\lambda]^{<\kappa} - \{0\}$ be such that $F(\beta, \alpha) \in a$ for all $\beta, \alpha \in a$ with $\beta \leq \alpha$. Then $f_{n+1}[[a]^{n+1}] \subseteq a$ for all $n \in [1, \omega)$, and $g_i[[a]^2] \subseteq a$ for all $i < 2$. Moreover, if ν is the least infinite cardinal $\leq \kappa$ with $a \cap \nu \neq \nu$, then ν is a successor and $a \cap \nu \in \nu$. Also note that if ρ is an uncountable cardinal $< \kappa$ with o.t. $(a \cap \rho^+) > \rho$, then $|a \cap \rho| = \rho$.

Let us finally assume that for every successor cardinal $\nu \in \cup(a \cap \kappa)$, there exists $n < \omega$ such that $\nu^{+n} \in \cup(a \cap \kappa)$ and $|a \cap \nu^{+n}| = \nu^{+n}$. Then $a \cap \kappa \in \kappa$. It is easily verified that $a = \cup\{a_d\colon d \in [a]^{<\omega} - \{0\}\}$. As D is closed under directed unions of size $< \kappa$, we have $a \in D$. ∎

Let $\varphi(x, p_0, \ldots, p_k)$ be a formula of set theory with parameters $p_0, \ldots, p_k$. We let $\pi_{\kappa,\lambda}(\varphi(x, p_0, \ldots, p_k))$ mean that given $D \in NS^*_{\kappa,\lambda}$, there exists F: $\lambda \cup [\lambda]^2 \to \lambda$ with the property that $a \in D$ for all $a \in [\lambda]^{<\kappa} - \{0\}$ such that $F[a \cup [a]^2] \subseteq a$ and $\varphi(a, p_0, \ldots, p_k)$. By Proposition 3.1, we have the following :

(i) $\pi_{\kappa,\lambda}(x = x)$ for $\kappa = \omega_1$.

(ii) $\pi_{\kappa,\lambda}(|x \cap \omega_n| = \aleph_n)$ in case $n \in (0, \omega)$ and $\kappa = \omega_{n+1}$.

(iii) $\pi_{\kappa,\lambda}(\{n \in \omega\colon |x \cap \omega_n| = \aleph_n\} \in [\omega]^\omega)$ for $\kappa = \aleph_{\omega+1}$.

(iv) $\pi_{\kappa,\lambda}(\nu \subseteq x)$ in case $\kappa = \nu^+$.

Given an infinite cardinal ν, we say that there is a Jonsson algebra on ν if there exists $g\colon \nu \times \nu \to \nu$ such that $g[b \times b] - b \neq 0$ for all $b \in [\nu]^\nu - \{\nu\}$. See [4] and [3] for more on Jonsson algebras.

PROPOSITION 3.2: *Assume that $\kappa = \nu^+$, and that there is a Jonsson algebra on ν. Then $\pi_{\kappa,\lambda}(|x \cap \nu| = \nu)$ holds.*

Proof: Choose $g\colon \nu \times \nu \to \nu$ such that $g[b \times b] - b \neq 0$ for every $b \in [\nu]^\nu - \{\nu\}$. Fix $D \in NS^*_{\kappa,\lambda}$, and let j be as in the proof of Proposition 3.1. Now define $F\colon \{(\beta, \alpha) \in \lambda \times \lambda\colon \beta \leq \alpha\} \to \lambda$ so that F satisfies conditions (7)–(10) of Proposition 3.1, and moreover for every $m \in \omega$ and every limit ordinal δ,

(i) $F(\alpha, \delta + 3^{m+1}) = j(\alpha, \delta + m)$ in case $\delta + m > \alpha$;

(ii) $F(\alpha, \delta + 5^{m+1}) = g(\alpha, \delta + m)$ in case $\delta + m \in [\alpha, \nu)$;

(iii) $F(\alpha, \delta + 7^{m+1}) = g(\delta + m, \alpha)$ in case $\delta + m \in (\alpha, \nu)$.

Then proceed as in the proof of Proposition 3.1, and observe the following. If $a \in [\lambda]^{<\kappa}$ is such that $|a \cap \nu| = \nu$, and that $F(\beta, \alpha) \in a$ for all $\beta, \alpha \in a$ with $\beta \leq \alpha$, then $\nu \subseteq a$. ∎

See Proposition 4.3 for a partial converse to Proposition 3.2.

Let $\mu \leq \lambda$ be a cardinal, and let $F\colon \mu \times \lambda \to \lambda$. We let $C_{F,\kappa}$ denote the set of all $a \in [\lambda]^{<\kappa}$ such that $a \cap \kappa \in \kappa$ and $F[(a \cap \mu) \times a] \subseteq a$. For every $b \in [\lambda]^{<\kappa}$, we set $e(F, b, \kappa) = \bigcup_{n\in\omega} b_n$, where

(0) $b_0 = b \cup F[(b \cap \mu) \times b]$;

(1) $b_{2p+1} = b_{2p} \cup F[(b_{2p} \cap \mu) \times b_{2p}]$;

(2) $b_{2p+2} = b_{2p+1} \cup \bigcup(b_{2p+1} \cap \kappa)$.

Clearly, $e(F, b, \kappa) = \bigcap\{a \in C_{F,\kappa}\colon b \subseteq a\}$.

PROPOSITION 3.3: *Assume $\lambda > \kappa$. Then there is $D \in NS^*_{\kappa,\lambda}$ such that for every regular cardinal $\mu < \lambda$ and every $F\colon \mu \times \lambda \to \lambda$, $C_{F,\kappa} - D$ is unbounded in $[\lambda]^{<\kappa}$.*

Proof: Choose a one-to-one function $j\colon [\lambda]^2 \to \lambda$, and let D be the set of all $a \in [\lambda]^{<\kappa}$ such that $j[[a]^2] = a \cap \operatorname{ran}(j)$. Let $F\colon \mu \times \lambda \to \lambda$ be given, where μ is a regular cardinal $< \lambda$, and fix $d \in [\lambda]^{<\kappa}$. Set $\nu = \kappa \cup \mu$. Choose $S \in [\nu^+]^{\nu^+}$ such that $e(F, d \cup \{\alpha\}, \nu) \cap \nu = e(F, d \cup \{\beta\}, \nu) \cap \nu$ for all $\alpha, \beta \in S$. Pick $X \in [S]^\nu$ and $\gamma_0 \in S - \bigcup_{\alpha\in X} e(F, d \cup \{\alpha\}, \nu)$. Then find $\gamma_1 \in X - e(F, d \cup \{\gamma_0\}, \nu)$. For each $i < 2$, set $x_i = \bigcup_{n\in\omega} x_i^n$, where

(0) $x_0^i = d \cup \{\gamma_i\} \cup F[(d \cap \mu) \times (d \cup \{\gamma_i\})]$;

(1) $x_i^{3p+1} = x_i^{3p} \cup \bigcup(x_i^{3p} \cap \kappa)$;
(2) $x_i^{3p+2} = x_i^{3p+1} \cup (x_{1-i}^{3p+1} \cap \mu)$;
(3) $x_i^{3p+3} = x_i^{3p+2} \cup F[(x_i^{3p+2} \cap \mu) \times x_i^{3p+2}]$.

Clearly $\gamma_i \in x_i - x_{1-i}$. Moreover $x_0 \cap \mu = x_1 \cap \mu$. Finally set $x_2 = x_0 \cup x_1$. Observe that for all $m < 3$, $d \subseteq x_m$ and $x_m \in C_{F,\kappa}$. It is not difficult to verify that $\{x_0, x_1, x_2\} - D \neq 0$. ∎

Let $c(\kappa, \lambda)$ be the smallest cardinality of any closed unbounded subset of $[\lambda]^{<\kappa}$.

We of course have $s(\kappa, \lambda) \leq c(\kappa, \lambda) \leq \lambda^{<\kappa}$. The following is a special case of Proposition 1.8 of [17].

Proposition 3.4: *Let $\mu \in [\kappa, \lambda]$ be a cardinal, let F: $\lambda \times \lambda \to \lambda$, and set $D = \{b \cap \mu : b \in C_{F,\kappa}\}$. Then D is a closed unbounded subset of $[\mu]^{<\kappa}$.*

Proof: It is easily verified that $D = \{a \in [\mu]^{<\kappa} : e(F, a, \kappa) \cap \mu = a\}$. Now D is unbounded in $[\mu]^{<\kappa}$, since for every $a \in [\mu]^{<\kappa}$, $e(F, a, \kappa) \cap \mu \in D$. Then let $a_\alpha \in D$, $a < \gamma < \kappa$, be such that $a_\beta \subseteq a_\alpha$ for $\beta < \alpha$. We have $e(F, \bigcup_{\alpha<\gamma} a_\alpha, \kappa) = \bigcup_{\alpha<\gamma} e(F, a_\alpha, \kappa)$, and consequently $\bigcup_{\alpha<\gamma} a_\alpha \in D$. ∎

Corollary 3.5: *$c(\kappa, \mu) \leq c(\kappa, \lambda)$ for each cardinal $\mu \in [\kappa, \lambda]$.*

The following is a slight improvement upon Theorem 1.1 of [2].

Proposition 3.6: *Let $n < \omega$, and let $\mu_i \in [\kappa, \lambda]$, $i \leq n$, be a strictly decreasing sequence of regular cardinals. For each i, let $S_i \subseteq \{\alpha : \mathrm{cof}(\alpha) < \kappa\}$ be a stationary subset of μ_i, and let $X_i \in [(\kappa, \lambda]]^{<\kappa}$ consist of cardinals of cofinality $\geq \kappa$. Assume that $X_0 \subseteq (\mu_0, \lambda]$ and that $X_{j+1} \subseteq (\mu_{j+1}, \mu_j)$. Then $T \in NS^+_{\kappa,\lambda}$, where T is the set of all $a \in [\lambda]^{<\kappa}$ such that for each $i \leq n$, $\bigcup(a \cap \mu_i) \in S_i$ and for all $\nu \in X_i$, $\mathrm{cof}(\bigcup(a \cap \nu)) = \mathrm{cof}(\bigcup(a \cap \mu_i))$.*

Proof: Wlog assume that $\mu_n = \kappa$. Fix F: $\lambda \times \lambda \to \lambda$. Define for each i, $a_i^\alpha \in [\lambda]^{<\mu_i}$, $\alpha < \mu_i$, $\alpha_i \in \mu_i$ and φ_i: $X_i \cup \{\mu_i\} \to [a_i^{\alpha_i}]^{<\kappa}$ so that

(0) o.t. $\varphi_i(\nu) = \mathrm{cof}(\bigcup(a_i^{\alpha_i} \cap \nu))$;
(1) $\bigcup\varphi_i(\nu) = \bigcup(a_i^{\alpha_i} \cap \nu)$;
(2) $a_i^0 = \bigcup_{j<i} \mathrm{ran}(\varphi_j) \cup \bigcup_{i<k} X_k \cup \{\mu_k\}$;
(3) $a_i^\alpha = \bigcup_{\beta<\alpha} a_i^\beta$ whenever $\alpha > 0$ is a limit ordinal ;
(4) $a_i^{\alpha+1} = a_i^\alpha \cup F[a_i^\alpha \times a_i^\alpha] \cup \bigcup(a_i^\alpha \cap \mu_i) \cup \{(\bigcup(a_i^\alpha \cap \nu)) + 1 : \nu \in X_i \cup \{\mu_i\}\}$;
(5) α_i is a limit ordinal;
(6) $\bigcup(a_i^{\alpha_i} \cap \mu_i) \in S_i$.

It is easily checked that $a_n^{\alpha_n} \in T \cap C_{F,\kappa}$. ∎

The following should be compared with Corollary 2.4 of [2].

PROPOSITION 3.7: *Let $n < \omega$, and let $\mu_i \in [\kappa, \lambda]$, $i \leq n+1$, be a strictly decreasing sequence of regular cardinals. Fix $q \in n+1$, and for each $i \neq q$, let $S_i \subseteq \{\alpha: \mathrm{cof}(\alpha) < \kappa\}$ be a stationary subset of μ_i. Put*

$$Y = \{a \in [\lambda]^{<\kappa}: \forall i \neq q \cup (a \cap \mu_i) \in S_i\}.$$

Then $|\{a \in Y \cap D: \mathrm{cof}(\cup(a \cap \mu_q)) = \omega\}| \geq \mu_q^{\aleph_0}$ for every $D \in NS^*_{\kappa,\lambda}$.

Proof: Wlog assume that $\mu_{n+1} = \kappa$. Fix $F: \lambda \times \lambda \to \lambda$. Use Proposition 3.6 to find $A \in [\lambda]^{<\mu_q^+}$ with the following properties : $F[A \times A] \subseteq A$, $A \cap \mu_q^+ \in \mu_q^+ - \mu_q$, and for every $i < q$, $\cup(A \cap \mu_i) \in S_i$. For each $i < q$, choose $z_i \in [A]^{<\kappa}$ with $\cup z_i = \cup(A \cap \mu_i)$. Then set $z = \bigcup_{i<q} z_i$. Put $T = \{\alpha \in (\mu_{q+1}, \mu_q): \mathrm{cof}(\alpha) = \omega\}$. For every $\alpha \in T$, inductively define $\beta_i^\alpha \in S_i$, $q < i \leq n+1$, so that

(0) $\beta_j^\alpha > \mu_{j+1}$ whenever $j < n+1$;

(1) $\mu_i \cap e(F, z \cup \mathrm{ran}(\tilde{\alpha}) \cup \bigcup_{q<j\leq i} \mathrm{ran}(\tilde{\beta_j^\alpha}), \mu_i) = \beta_i^\alpha$.

Choose β_i, $q < i \leq n+1$, and a stationary subset T_0 of T such that $\beta_i^\alpha = \beta_i$ whenever $\alpha \in T_0$ and $q < i \leq n+1$. Set $u = z \cup \beta_{n+1} \cup \bigcup_{q<i<n+1} \mathrm{ran}(\tilde{\beta}_i)$.

By induction on the domain of f, we define a stationary subset T_f of T_0 and $\gamma_f \in \mu_q$, $f \in \bigcup_{m\in\omega} \mu_q^{m+1}$, as follows. Let $g \in \mu_q^m$ be given. By induction on $\delta < \mu_q$, construct η_δ and Y_δ so that

(2) Y_δ is a stationary subset of T_g;

(3) $\eta_\delta \in \bigcap_{\alpha\in Y_\delta} \mathrm{ran}(\tilde{\alpha})$;

(4) $\eta_\delta > \cup(e(F, u \cup \{\gamma_{g|p}: 1 < p \leq m\}, \kappa) \cap \mu_q)$;

(5) $\delta' < \delta$ implies $\eta_{\delta'} < \eta_\delta$.

For each $\delta < \mu_q$ with $\mathrm{cof}(\delta) \geq \kappa$, pick $\zeta_\delta < \delta$ and a stationary subset W_δ of Y_δ such that for every $\alpha \in W_\delta$, $\{\eta_\xi: \zeta_\delta \leq \xi < \delta\} \cap e(F, u \cup \mathrm{ran}(\tilde{\alpha}), \kappa) = 0$. Select ζ and a stationary $B \subseteq \{\delta \in (\zeta, \mu_q): \mathrm{cof}(\delta) \geq \kappa\}$ such that $\zeta_\delta = \zeta$ for all $\delta \in B$. Finally let δ_ξ, $\xi < \mu_q$, be the increasing enumeration of B, and set $T_{g\cup\{(m,\xi)\}} = W_{\delta_\xi}$ and $\gamma_{g\cup\{(m,\xi)\}} = \eta_{\delta_\xi}$.

For each $f \in \mu_q^\omega$, set $x_f = e(F, u \cup \{\gamma_{f|m}: 0 < m < \omega\}, \kappa)$. Clearly,

$$e(F, u \cup \{\gamma_{f|m}: 0 < m < p\}, \kappa) \subseteq \bigcap_{\alpha\in T_{f|p}} e(F, u \cup \mathrm{ran}(\tilde{\alpha}), \kappa) \quad \text{for every } p > 1.$$

Consequently $x_f \in Y$. Also, $x_f \in C_{F,\kappa}$ and $\cup(x_f \cap \mu_q) = \bigcup_{0<m<\omega} \gamma_{f|m}$. Now let $f, g \in \mu_q^\omega$ and $m \in \omega$ be such that $f|m = g|m$ and $f(m) < g(m)$. Then $\gamma_{f|m+1} \notin$

$\bigcup_{\alpha\in T_{g|p}} e(F, u \cup \mathrm{ran}(\tilde{\alpha}), \kappa)$ for all $p > m$, and consequently $\gamma_{f|m+1} \in x_f - x_g$. ∎

The following is due to Baumgartner [2].

COROLLARY 3.8: *Assume $\lambda > \kappa$. Then $c(\kappa, \lambda) \geq \lambda^{\aleph_0}$.*

Proof: If $\kappa^{\aleph_0} \geq \lambda$, then the result follows from Proposition 3.7, as $\lambda^{\aleph_0} = \kappa^{\aleph_0}$. Otherwise, we have $u(\omega_1, \kappa) < \lambda$ and by Proposition 2.3, $u(\omega_1, \lambda) = \lambda^{\aleph_0}$. Then $\lambda^{\aleph_0} \leq u(\kappa, \lambda)$, as by Proposition 2.5, $u(\omega_1, \lambda) \leq u(\omega_1, \kappa) \cdot u(\kappa, \lambda)$. ∎

By a result of Magidor [13], if there is no ω_1-Erdös cardinal in the core model K, then $c(\kappa, \lambda) = \lambda^{\aleph_0}$ in case $\mathrm{cof}(\lambda) \geq \kappa$, and $c(\kappa, \lambda) = \lambda^+ \cdot \lambda^{\aleph_0}$ otherwise. On the other hand, Baumgartner [2] showed that it is consistent relative to the existence of an ω_1-Erdös cardinal, that $c(\omega_2, \omega_3) = \aleph_3^{\aleph_1}$ and $\aleph_3^{\aleph_0} < \aleph_3^{\aleph_1}$. Magidor [13] also showed the following. Assume either that there is no inner model with a measurable cardinal, and that $\lambda < \aleph_{\omega_2}$, or else that there is no ω_2 -Erdös cardinal in K. Then $c(\kappa, \lambda) \leq \lambda^{\aleph_1}$ in case $\mathrm{cof}(\lambda) \geq \kappa$, and $c(\kappa, \lambda) \leq \lambda^+ \cdot \lambda^{\aleph_1}$ otherwise.

4. Small stationary sets

We define $C_{\kappa,\lambda} \subseteq [\lambda]^{<\kappa}$ by letting $a \in C_{\kappa,\lambda}$ iff $0 \in a \cap \kappa \in \kappa$ and for all $\alpha \in a$, $\alpha + 1 \in a$ and $a \cap \alpha = \hat{\alpha}[a \cap |\alpha|]$.

It is easily seen that $C_{\kappa,\lambda} \in NS^*_{\kappa,\lambda}$. $C_{\kappa,\lambda}$ has the following interesting property.

PROPOSITION 4.1: *Let $\mu \in [\kappa, \lambda]$ be a cardinal, and let a, $d \in C_{\kappa,\lambda}$ with $a \cap \mu \neq d \cap \mu$. Then $\{b \subseteq a \cap d \cap \mu \colon \forall \alpha \in (a \cup d) \cap \mu \; b \cap (\alpha, \kappa \cdot |\alpha|^+) \neq 0\} = 0$.*

Proof: Let $a \in C_{\kappa,\lambda}$ and $b \subseteq a \cap \mu$ be given such that $b \cap (\alpha, \kappa \cdot |\alpha|^+) \neq 0$ for all $\alpha \in a \cap \mu$. Given $\beta < \mu$, define β_n, $n < \omega$, as follows. Set $\beta_0 = \beta$. Let $\beta_{n+1} = 0$ in case $\beta_n < a \cap \kappa$, and let $\beta_{n+1} = \lambda$ in case $(\beta_n, \kappa \cdot |\beta_n|^+) \cap a = 0$. If $(\beta_n, |\beta_n|^+) \cap (a - \kappa) \neq 0$, put $\beta_{n+1} = \hat{\alpha}^{-1}(\beta_n)$, where α is least in $b - (\beta_n + 1)$. Then it is easily seen that $\beta \in a$ iff $\beta_n = 0$ for some n. ∎

COROLLARY 4.2: *$s(\kappa, \kappa^{+\delta}) \leq (\kappa^{+\delta})^{|\delta|}$ for all $\delta \in \kappa - \{0\}$.*

Proof: By Proposition 2.1 and Proposition 2.5, $s(\kappa, \kappa^{+n}) = \kappa^{+n}$ for all $n \in \omega$. Now let $\delta \in \kappa - \omega$, and let S be the set of all $a \in C_{\kappa,\kappa^{+\delta}}$ such that $\mathrm{cof}(\bigcup(a \cap \kappa^{+(\beta+1)})) = \omega$ for all $\beta < \delta$. Then $S \in NS^+_{\kappa,\kappa^{+\delta}}$ and $|S| \leq (\kappa^{+\delta})^{|\delta|}$. ∎

We now briefly return to the problem of finding a more economical characterization of members of the closed unbounded filter.

PROPOSITION 4.3: *Assume that $\lambda < \kappa^{+\omega}$, that $\kappa = \nu^+$, and that $\pi_{\kappa,\lambda}(|x \cap \nu| = \nu)$ holds. Then there is a Jonsson algebra on ν.*

Proof: By the assumption, there is $F: \lambda \cup [\lambda]^2 \to \lambda$ such that $a \in C_{\kappa,\lambda}$ for all $a \in [\lambda]^{<\kappa}$ with $F[a \cup [a]^2] \subseteq a$ and $|a \cap \nu| = \nu$. Now pick $b \in [\lambda]^{<\kappa}$ so that $\nu \cup F[b \cup [b]^2] \subseteq b$, and $\mathrm{cof}(\bigcup(b)) = \omega$ for every cardinal $\mu \in [\kappa, \lambda]$. Define $h: \omega \to b$ so that $\bigcup(\mathrm{ran}(h) \cap \mu) = \bigcup(b \cap \mu)$ for every $\mu \in [\kappa, \lambda]$. Then let $g: b \times b \to b$ be such that

(0) $g(\alpha, \alpha) = \alpha + 1$;
(1) $g(\alpha + 1, \alpha) = 0$;
(2) $g(\beta + 2, \alpha) = \hat{\beta}^{-1}(\alpha)$ whenever $\beta > \alpha \geq \omega$;
(3) $g(n + 3, 0) = h(n)$ for $n \in \omega$;
(4) $g(\alpha + 2, \alpha) = F(\alpha)$;
(5) $g(\beta, \alpha) = F(\{\beta, \alpha\})$ for $\beta < \alpha$.

Now let $a \in [b]^\nu$ be such that $g[a \times a] \subseteq a$.

We claim that $|a \cap \nu| = \nu$. The claim is immediate in case $\nu = \omega$, as $\omega \subseteq a$. Thus assume $\nu > \omega$. It is clearly enough to show that $|a \cap \mu| = \nu$ whenever μ is a cardinal such that $\mu \geq \nu$ and $|a \cap (\mu^+ - \mu)| = \nu$. Let such a μ be given. Let us first assume that o.t. $(a \cap (\mu^+ - \mu)) > \nu$. Let $\beta \in a \cap (\mu^+ - \mu)$ be such that o.t. $(a \cap \beta) = \nu$. Then clearly $\hat{\beta}^{-1}[(a \cap \beta) - \omega] \in [a \cap \mu]^\nu$. Now consider the case when o.t. $(a \cap (\mu^+ - \mu)) = \nu$. Then ν is a limit cardinal (of cofinality ω), as $\bigcup(a \cap \mu^+) = \bigcup(b \cap \mu^+)$. Moreover, by the same argument as above, we have that $|a \cap \mu| \geq \rho$ for every cardinal $\rho < \nu$. Hence $|a \cap \mu| = \nu$.

It easily follows from the claim that $a = b$. Finally define $k: \nu \times \nu \to \nu$ by letting $k(\alpha, \beta) = \hat{b}^{-1}(g(\hat{b}(\alpha), \hat{b}(\beta)))$. It is easily verified that $k[a \times a] - a \neq 0$ for all $a \in [\nu]^\nu - \{\nu\}$. ∎

PROPOSITION 4.4: *Let $n < \omega$, let $q \in n + 1$, and for each $i \neq q$, let $S_i \subseteq \{\alpha: \mathrm{cof}(\alpha) < \kappa\}$ be a stationary subset of $\kappa^{+(n+1-i)}$. Put*

$$Y = \{a \in [\kappa^{+(n+1)}]^{<\kappa}: \forall i \neq q \bigcup (a \cap \kappa^{+(n+1-i)}) \in S_i\}.$$

Let $D \in NS^*_{\kappa,\kappa^{+(n+1)}}$, and set $Z = \{a \in Y \cap D: \mathrm{cof}(\bigcup(a \cap \kappa^{+(n+1-q)})) = \omega\}$. Then there are $x_f \in Z$, $f \in (\kappa^{+(n+1-q)})^\omega$, such that for all f, $f' \in (\kappa^{+(n+1-q)})^\omega$ with $f \neq f'$, $x_f - x_{f'} \neq 0$ and $x_{f'} - x_f \neq 0$.

Proof: Pick $F\colon \kappa^{+(n+1)} \times \kappa^{+(n+1)} \to \kappa^{+(n+1)}$ with $C_{F,\kappa} \subseteq D \cap C_{\kappa,\kappa^{+(n+1)}}$. Let T_0, u and β_i, $q < i \le n+1$, be as in the proof of Proposition 3.7. Define d_i, $i \in (q, n+1]$, by letting $d_{n+1} = \beta_{n+1}$ and for all $j \in (q, n+1)$, $d_j = \bigcup_{\alpha \in \mathrm{ran}(\bar{\beta_j})} \hat{\alpha}[d_{j+1}]$. Now for all $f \in \bigcup_{m\in\omega}(\kappa^{+(n+1-q)})^m$, define a stationary subset T_f of T_0 and $\gamma_f \in \kappa^{+(n+1-q)}$ as follows. Let $g \in (\kappa^{+(n+1-q)})^m$ be given. Select η_δ and Y_δ, $\delta < \kappa^{+(n+1-q)}$, so that

(i) conditions (2)–(5) of the proof of Proposition 3.7 are satisfied;
(ii) $\eta_\delta \ge \kappa^{+(n-q)}$;
(iii) o.t. $\hat{\eta_\delta}[d_{q+1}] =$ o.t. $\hat{\eta_{\delta'}}[d_{q+1}]$ for all δ, $\delta' < \kappa^{+(n+1-q)}$.

Then set $T_{g\cup\{(m,\delta)\}} = Y_\delta$ and $\gamma_{g\cup\{(m,\delta)\}} = \eta_\delta$.

Finally for each $f \in (\kappa^{+(n+1-q)})^\omega$, set $x_f = e(F, u \cup \{\gamma_{f|m}\colon 0 < m < \omega\}, \kappa)$. Each $x_f \in Z \cap C_{\kappa,\kappa^{+(n+1)}}$. Now let f, $g \in (\kappa^{+(n+1-q)})^\omega$ and $m \in \omega$ be such that $f|m = g|m$ and $f(m) \neq g(m)$. Then $x_f \cap \gamma_{f|m} = x_g \cap \gamma_{f|m}$, but $\gamma_{f|m+1} \in x_f - x_g$ and $\gamma_{g|m+1} \in x_g - x_f$, as o.t. $(x_f \cap \gamma_{f|m+1}) =$ o.t. $(x_g \cap \gamma_{g|m+1})$. ∎

Let A be a set of ordinals, and let ρ be a cardinal with o.t. $A > \rho$. A is said to be ρ-closed if $\bigcup d \in A$ for all $d \subset A$ with o.t. $d = \rho$.

Let $R_{\kappa,\lambda}$ be the set of those $a \in [\lambda]^{<\kappa}$ that satisfy the following conditions:

(0) $\alpha + 1 \in a$ iff $\alpha \in a$;
(1) $a \cap \kappa \in \kappa$;
(2) given a limit $\alpha \in a$, $\tilde{\alpha}(\beta) \in a$ iff $\beta \in a$, and for every $\gamma \in a$, $\bigcup(\gamma \cap \mathrm{ran}(\tilde{\alpha})) \in a$.

$R_{\kappa,\lambda} \in NS^*_{\kappa,\lambda}$. Moreover the following holds.

PROPOSITION 4.5: *Let $\mu \in [\kappa, \lambda]$ be a cardinal, let $a \in R_{\kappa,\lambda}$ and let $\rho \in [\omega, \kappa)$ be a regular cardinal such that $\rho \neq \mathrm{cof}(\bigcup(a \cap (\nu \cap \mu)))$ for every regular cardinal $\nu \in [\kappa, \mu^+]$. Then $a \cap \mu$ is ρ-closed.*

Proof: Let $d \subset a \cap \mu$ be such that o.t. $d = \rho$. We have $\bigcup d < \bigcup(a \cap \mu)$, as $\mathrm{cof}(\bigcup(a \cap \mu)) \neq \rho$. Set $\alpha = \bigcap(a - \bigcup d)$. Then α is a limit ordinal. It is readily verified that $\bigcup d = \bigcup_{\beta \in a \cap \mathrm{cof}(\alpha)} \tilde{\alpha}(\beta)$. We cannot have $\mathrm{cof}(\alpha) > a \cap \kappa$, as this would yield $\mathrm{cof}(\bigcup(a \cap (\kappa \cdot \mathrm{cof}(\alpha))) = \rho$. Thus $a \cap \mathrm{cof}(\alpha) = \mathrm{cof}(\alpha)$, and consequently $\bigcup d = \alpha$. ∎

The following is due to Baumgartner [2].

COROLLARY 4.6: *$c(\kappa, \kappa^{+n}) \le \kappa^{\aleph_n} \cdot \kappa^{+n}$ for all $n < \omega$.*

Proof: By Proposition 4.1 and Proposition 4.5. ∎

We will devote the remainder of the section to a generalization of Lemma 3.6 of [2].

Let $n \in \omega$, and let $\mu \in [\kappa^{+(n+1)}, \lambda]$ be a cardinal. Also let ρ_i, $i \leq n+1$, be a strictly decreasing sequence of regular cardinals $< \kappa$. For every $a \in [\lambda]^{<\kappa}$, we define the two-person game $G(a)$ as follows. Each player makes $n+1$ moves. I (respectively II) produces y_i (resp. z_i), $i \leq n$, such that

(0) $y_0 \subseteq a \cap \mu$ and $y_{j+1} \subseteq z_j$;

(1) $z_i \subseteq y_i$;

(2) o.t. $y_i =$ o.t. $z_i = \rho_i$;

(3) $\cap(a - \cup d) \in y_0$ for every $d \subseteq y_0$ with o.t. $d = \rho_{n+1}$;

(4) $\cap(z_j - \cup d) \in y_{j+1}$ for every $d \subseteq y_{j+1}$ with o.t. $d = \rho_{n+1}$.

II wins iff z_n is ρ_{n+1}-closed.

PROPOSITION 4.7: *Let $a \in R_{\kappa,\lambda}$ with $\text{cof}(\cup(a \cap \mu)) \neq \rho_0$ and $\rho_0 \subseteq a \cap \kappa$, and assume that* II *has no winning strategy in $G(a)$. Then there is a strictly decreasing sequence ν_i, $i \leq n+1$, of regular cardinals such that $\nu_{n+1} \geq \kappa$, $\nu_0 < \mu$ and for all i, $\text{cof}(\cup(a \cap \nu_i)) = \rho_i$.*

Proof: Assume that the conclusion of the proposition fails. We will define a winning strategy τ for II in $G(a)$. Consider a run of the game where I plays y_i, $i \leq n$. Inductively define d_i, α_i, c_i and φ_i, $i \leq n$, so that

(0) $d_0 = y_0$;

(1) $\alpha_i = \cap(a - \cup d_i)$;

(2) $c_i \subseteq \text{cof}(\alpha_i)$ and o.t. $c_i = \rho_i$;

(3) φ_i is a strictly increasing function from c_i to d_i such that for every $\gamma \in c_i$, $\varphi_i(\gamma) = \cap(d_i - \tilde{\alpha}(\gamma))$ and $\tilde{\alpha}_i(\gamma) = \cap(\text{ran}(\tilde{\alpha}_i) - \bigcup_{\delta \in \gamma \cap c_i}(\varphi_i(\delta) + 1))$;

(4) $\tau(y_0, ..., y_i) = \varphi_0[\varphi_1[...[\varphi_i[c_i]]...]]$;

(5) $d_{j+1} \subset c_j$, and $y_{j+1} = \varphi_0[\varphi_1[...[\varphi_j[d_{j+1}]]...]]$.

Finally let $e \subseteq \tau(y_0, ..., y_n)$ with o.t. $e = \rho_{n+1}$. Let u be such that $e = \varphi_0[\varphi_1[...[\varphi_n[u]]...]]$. Then $\cup u \in a$, and consequently $\cup e \in a$, as

$$\cup e = \tilde{\alpha_0}(\tilde{\alpha_1}(...(\tilde{\alpha_n}(\cup u))...)).$$

It is now easy to check that $\cup e \in \tau(y_0, ..., y_n)$. ∎

Inductively define the $(\rho_0, ..., \rho_i)$-filter on a limit ordinal α with $\text{cof}(\alpha) > \rho_0$ as follows :

(0) $A \subseteq \alpha$ lies in the (ρ_0)-filter on α iff A contains a set B such that B is ρ_0-closed and unbounded in α;

(1) $A \subseteq \alpha$ lies in the $(\rho_0, ..., \rho_{j+1})$-filter on α iff $\{\beta < \alpha$: $A \cap \beta$ lies in the $(\rho_1, ..., \rho_{j+1})$-filter on $\beta\}$ lies in the (ρ_0)-filter on α.

We leave it to the reader to verify that the $(\rho_0, ..., \rho_i)$-filter on α is a ρ_i^+-complete filter, and that each of its members is cofinal in α.

PROPOSITION 4.8: *Let $a \in [\lambda]^{<\kappa}$ with $\mathrm{cof}(\bigcup(a \cap \mu)) > \rho_0$, and assume II has a winning strategy in $G(a)$. Then $a \cap \mu$ lies in the $(\rho_0, ..., \rho_{n+1})$-filter on $\bigcup(a \cap \mu)$.*

Proof: Let A be the set of all $\beta < \bigcup(a \cap \mu)$ such that there is $d \subseteq a$ with o.t. $d = \rho_0$ and $\bigcup d = \beta$. It is not difficult to show that A lies in the $(\rho_1, ..., \rho_{n+1})$-filter on $\bigcup(a \cap \mu)$. ∎

5. U_ν^h

Let h: $\lambda \to [\lambda]^{<\kappa}$, and let $\nu < \kappa$ be an infinite cardinal. We let U_ν^h denote the set of all $a \in [\lambda]^{<\kappa}$ such that there exists $d \in [a]^\nu$ with $a = \bigcup_{\alpha \in d} h(\alpha)$.

Notice that if $h(\alpha) = \{\alpha\}$, then $a \in U_{|a|}^h$ for all infinite $a \in [\lambda]^{<\kappa}$. If $\lambda = \kappa$ and $h(\alpha) = \alpha$, then $\alpha \in U_{\mathrm{cof}(\alpha)}^h$ for every limit ordinal $\alpha \in (0, \kappa)$.

Set $B(\kappa, \lambda) = \{a \in [\lambda]^{<\kappa}$: $\bigcup a \in \lambda\}$ and

$$UB(\kappa, \lambda) = \{E \subseteq B(\kappa, \lambda)\colon B(\kappa, \lambda) = \bigcup_{a \in E} P(a)\}.$$

Using the results of Section 2, it is easy to see that $|E| = \lambda$ for some $E \in UB(\kappa, \lambda)$ iff $u(\kappa, \mu) \leq \lambda$ for every cardinal $\mu \in [\kappa, \lambda)$.

PROPOSITION 5.1: *Assume that* $\mathrm{ran}(h) \in UB(\kappa, \lambda)$. *Then* $U_\nu^h \Delta U_\nu^{h'} \in NS_{\kappa,\lambda}$ *for every* h': $\lambda \to [\lambda]^{<\kappa}$ *with* $\mathrm{ran}(h') \in UB(\kappa, \lambda)$.

Proof: Let F: $\lambda \to \lambda$ be such that $h(\alpha) \subseteq h'(F(\alpha))$. Then

$$\{a \in U_\nu^h\colon F[a] \cup \bigcup_{\beta \in a} h'(\beta) \subseteq a\} \subseteq U_\nu^{h'}. \quad ∎$$

Let us observe the following. For each $a \in U_\nu^h$, pick $d_a \in [a]^\nu$ with $a = \bigcup_{\alpha \in d_a} h(\alpha)$. Put $S = \{a \in U_\nu^h\colon \bigcup d_a < \bigcup a\}$. Then $S \in NS_{\kappa,\lambda}$, since otherwise there would exist $S' \in P(S) \cap NS_{\kappa,\lambda}^+$ and $\alpha \in \lambda$ such that $\bigcup S' \subseteq \bigcup_{\beta < \alpha} h(\beta)$, which cannot be, as $|\bigcup_{\beta<\alpha} h(\beta)| < \lambda$.

PROPOSITION 5.2: $\{a \in U^h_\nu : |a| > |a \cap \kappa|\} \in NS_{\kappa,\lambda}$.

Proof: $\{a \in [\lambda]^{<\kappa} : \forall \alpha \in a\ |h(\alpha)| \leq |a \cap \kappa|\} \in NS^*_{\kappa,\lambda}$. ∎

PROPOSITION 5.3: *Let* $\mu \in [\kappa, \lambda]$ *be a cardinal with* $\mathrm{cof}(\mu) \geq \kappa$. *Then* $\{a \in U^h_\nu : \mathrm{cof}(\cup(a \cap \mu)) > \nu\} \in NS_{\kappa,\lambda}$.

Proof: $\{a \in [\lambda]^{<\kappa} : \forall \alpha \in a \cup (h(\alpha) \cap \mu) < \cup(a \cap \mu)\} \in NS^*_{\kappa,\lambda}$. ∎

PROPOSITION 5.4: *Assume that* $\mathrm{ran}(h) \in UB(\kappa, \lambda)$, *and that* $\mathrm{cof}(\lambda) \notin (\nu, \kappa)$. *Let* $n \in \omega$, *and let* $\mu_i \in [\kappa, \lambda]$, $i \leq n$, *be a strictly decreasing sequence of regular cardinals. For each* $i \leq n$, *let* $S_i \in NS^+_{\mu_i}$ *with* $S_i \subseteq \{\alpha : \mathrm{cof}(\alpha) \leq \nu\}$. *Then* $T \cap U^h_\nu \in NS^+_{\kappa,\lambda}$, *where* $T = \{a \in [\lambda]^{<\kappa} : \forall i \leq n \cup (a \cap \mu_i) \in S_i\}$.

Proof: Fix $D \in NS^*_{\kappa,\lambda}$. Choose $F: \lambda \times \lambda \to \lambda$ such that

$$C_{F,\kappa} \subseteq \{a \in D : a = \bigcup_{\alpha \in a} h(\alpha)\}.$$

We define $g: [\lambda]^{<\omega} \to C_{F,\kappa} \cap U^h_\nu$ as follows. Given $c \in [\lambda]^{<\omega}$, let a_m, b_m, c_m and d_m, $m < \omega$, be such that:

(i) $c_0 = c \cup \nu \cup \bigcup_{d \subset c} g(d)$;

(ii) $a_m = c_m \cup F[c_m \times c_m]$;

(iii) $b_m = \cup(a_m \cap \kappa)$;

(iv) $|d_m| \leq \nu$ and $a_m \cup b_m \subseteq \bigcup_{\alpha \in d_m} h(\alpha)$;

(v) $c_{m+1} = d_m \cup \bigcup_{\alpha \in d_m} h(\alpha)$.

Then set $g(c) = \bigcup_{m \in \omega} a_m$.

Now define by induction $\alpha_i \in S_i$, $i \leq n$, so that $g(c) \cap \mu_i \subseteq \alpha_i$ for all $c \in [\alpha_i \cup \bigcup_{j<i} \mathrm{ran}(\tilde{\alpha_j})]^{<\omega}$. Finally set $a = \cup\{g(c) : c \in [\bigcup_{i \leq n} \mathrm{ran}(\tilde{\alpha_i})]^{<\omega}\}$. Clearly $a \in C_{F,\kappa} \cap U^h_\nu$. Moreover $\cup(a \cap \mu_i) = \alpha_i$ for all $i \leq n$. ∎

PROPOSITION 5.5: *Assume that* $\mathrm{ran}(h) \in UB(\kappa, \lambda)$, *and let* $S \in NS^+_{\mu,\lambda}$, *where* $\mu \in (\omega, \kappa]$ *is a cardinal such that* $\mathrm{cof}(\lambda) \notin [\mu, \kappa)$. *Set*

$$D = \{a \in [\lambda]^{<\mu} : a \subseteq \bigcup_{\alpha \in a} h(\alpha)\} \quad \text{and} \quad T = \{\bigcup_{\alpha \in a} h(\alpha) : a \in S \cap D\}.$$

Then $T \in NS^+_{\kappa,\lambda}$.

Proof: Left to the reader. ∎

It is easy to see that Proposition 5.4 can be derived from Proposition 5.5, Proposition 3.6 and the following observation. Let $\eta \in [\kappa, \lambda]$ be a cardinal with $\mathrm{cof}(\eta) \geq \kappa$. Then $\{a \in [\lambda]^{<\mu} : \cup (a \cap \eta) = \cup((\bigcup_{\alpha \in a} h(\alpha)) \cap \eta)\} \in NS^*_{\mu,\lambda}$.

PROPOSITION 5.6: *Assume $\lambda = \kappa^{+\delta}$, where $|\delta| \leq \nu$, and let S be the set of all $a \in [\lambda]^{<\kappa}$ such that for every regular cardinal $\mu \in [\kappa, \lambda]$, $\mathrm{cof}(\bigcup(a \cap \mu)) \leq \nu$. Then $S \triangle U^h_\nu \in NS_{\kappa,\lambda}$ for some $h: \lambda \to [\lambda]^{<\kappa}$.*

Proof: Let $g: [\lambda]^{<\omega} \to [\lambda]^{<\kappa}$ be as follows:

(i) $g(\{\alpha\}) = \alpha$ for all $\alpha \in \kappa$;

(ii) let $n \in \omega - 1$, and let $\alpha_p \in \lambda$, $p \leq n$, be such that $\alpha_0 \in \kappa$ and for all $j < n$, $\alpha_j < |\alpha_{j+1}|$. Then $g(\{\alpha_p : p \leq n\}) = \hat{\alpha_n}[...[\hat{\alpha_1}[\alpha_0]]...]$.

Select a bijection $j: [\lambda]^{<\omega} \to \lambda$, and set $h = g \circ j^{-1}$. Then

$$\{a \in S \cap C_{\kappa,\lambda} : j[[a]^{<\omega}] = a\} \subseteq U^h_\nu.$$

Finally apply Proposition 5.3 to get $U^h_\nu - S \in NS_{\kappa,\lambda}$. ∎

Let us observe that if $\lambda = \kappa^{+\delta}$ with $\delta \leq \omega$, and if $h: \lambda \to [\lambda]^{<\kappa}$ is as in the proof of Proposition 5.6, then $\mathrm{ran}(h) \in UB(\kappa, \lambda)$.

PROPOSITION 5.7: *Let $\mu \in (\omega, \kappa)$ be a regular cardinal, and let $T \subseteq \{a \in \bigcup_{\omega \leq \eta < \mu} U^h_\eta : \mu \cup \bigcup_{\alpha \in a} h(\alpha) \subseteq a\}$. Then there exist pairwise disjoint $D_a \in NS^*_{\mu,a}$, $a \in T$.*

Proof: For each $a \in T$, select $d_a \subseteq a$ such that $|d_a| < \mu$ and $a = \bigcup_{\alpha \in d_a} h(\alpha)$. Now put $D_a = \{b \in [a]^{<\mu} : d_a \subseteq b\}$. ∎

6. $\diamondsuit_{\kappa,\lambda}$

For each $b \in [\lambda]^{<\kappa}$, let $\bar{b}$: o.t.$b \to b$ be the increasing enumeration of b. Given $a \in [\lambda]^{<\kappa} - \{0\}$ and an infinite cardinal $\mu < \kappa$, let $K_{a,\mu}$ denote the set of all $k \in P(a)^\mu$ such that $a = \bigcup_{\alpha<\mu} k(\alpha)$, and that for all $\alpha, \beta < \mu$ with $a \leq \beta$, $k(\alpha) \subseteq k(\beta)$. Then let $\varphi_{a,\mu}: K_{a,\mu} \to \kappa^\mu \times P(\kappa)^{[\mu]^2}$ be defined as follows. We let $\varphi_{a,\mu}(k) = (g, h)$, where $g(\alpha) = \text{o.t.}k(\alpha)$ and $h(\alpha, \beta) = \overline{k(\beta)}^{-1}[k(\alpha)]$.

LEMMA 6.1: *$\varphi_{a,\mu}$ is one-to-one.*

Proof: Fix $(g, h) \in \mathrm{ran}(\varphi_{a,\mu})$. Define r: (o.t.a) $\times \mu \to P(\kappa)$, s: (o.t.a) $\times \mu \to \kappa \cup \{\kappa\}$ and t: o.t.$a \to \mu$ so that

(0) $r(\gamma, \alpha) \subseteq g(\alpha)$;

(1) $r(0, \alpha) = 0$;

(2) $r(\gamma, \alpha) = \bigcup_{\delta<\gamma} r(\delta, \alpha)$ whenever γ is a limit ordinal with $\gamma > 0$;

(3) $s(\gamma, \alpha) = \kappa$ in case $r(\gamma, \alpha) = g(\alpha)$;

(4) $s(\gamma,\alpha) = \bigcap(g(\alpha) - r(\gamma,\alpha))$ in case $r(\gamma,\alpha) \subset g(\alpha)$;
(5) $s(\gamma, t(\gamma)) \notin h(\beta, t(\gamma))$ for all $\beta < t(\gamma)$;
(6) $s(\gamma,\eta) \in h(t(\gamma),\eta)$ for all $\eta > t(\gamma)$;
(7) $r(\gamma+1,\alpha) = r(\gamma,\alpha)$ for all $\alpha < t(\gamma)$;
(8) $r(\gamma+1,\alpha) = r(\gamma,\alpha) \cup \{s(\gamma,\alpha)\}$ for all $\alpha \geq t(\gamma)$.

Finally define $k \in K_{a,\mu}$ by letting $k(\alpha) = \{\overline{a}(\gamma)\colon \alpha \geq t(\gamma)\}$. It is easily verified that $\{k\} = \varphi_{a,\mu}^{-1}(g,h)$. ∎

PROPOSITION 6.2: *Let $\eta \in [\kappa,\lambda]$, ν and $\mu \in [0,\nu)$ be cardinals such that* $\mathrm{cof}(\eta) >$ μ, $\kappa = \nu^+$, $2^\nu \leq \lambda$, $\mathrm{cof}(\mu) = \mu$ *and* $\nu^\mu = \nu$. *Then*

$$\{a \in [\lambda]^{<\kappa}\colon \mathrm{cof}(\bigcup(a \cap \eta)) = \mu\} \in ND^+_{\kappa,\lambda}.$$

Proof: Select a bijection $j\colon [\kappa]^{<\kappa} \to 2^\nu$. Given $a \in [\lambda]^{<\kappa}$, we define $w_a \subseteq P(a)$ by letting $b \in w_a$ iff there exist $k \in K_{a,\mu}$ and $m\colon \mu \to a \cap 2^\nu$ such that, setting $(g,h) = \varphi_{a,\mu}(k)$, we have

(0) $\mathrm{ran}(g) \subseteq a \cap \kappa$;
(1) $j[\mathrm{ran}(h)] \subseteq a \cap 2^\nu$;
(2) $b = \bigcup_{\alpha<\mu} \overline{k(\alpha)}[j^{-1}(m(\alpha)) \cap \mathrm{o.t.}k(\alpha)]$.

Now fix $A \subseteq \lambda$ and $F\colon \lambda \times \lambda \to \lambda$. Define $k\colon \mu \to [\lambda]^{<\kappa}$ so that

(i) $\mathrm{o.t.}k(\alpha) \in k(\alpha+1)$;
(ii) $j(\overline{k(\beta)}^{-1}[k(\alpha)]) \in k(\beta+1)$ whenever $\alpha < \beta$;
(iii) $j(\overline{k(\alpha)}^{-1}[A \cap k(\alpha)]) \in k(\alpha+1)$;
(iv) $k(\alpha) \subseteq k(\alpha+1)$;
(v) $k(\beta) = \bigcup_{\alpha<\beta} k(\alpha)$ whenever β is a limit ordinal with $\beta > 0$;
(vi) $(\bigcup(k(\alpha) \cap \eta)) + 1 \in k(\alpha+1)$;
(vii) $k(\alpha) \in C_{F,\kappa}$.

Finally set $a = \bigcup_{\alpha<\mu} k(\alpha)$. Then $a \in C_{F,\kappa}$, $\mathrm{cof}(\bigcup(a \cap \eta)) = \mu$ and $A \cap a \in w_a$. ∎

7. $\diamondsuit_{\kappa,\lambda}$ via Cohen forcing

Given $S \subseteq [\lambda]^{<\kappa}$ and a cardinal $\mu \in [\kappa,\lambda]$ with $\mathrm{cof}(\mu) \geq \kappa$, $\diamondsuit_{\kappa\colon \mu\colon \lambda}(S)$ asserts the existence of a sequence $s_\alpha \in \bigcup_{\beta\in\mathrm{On}} P(\beta)$, $\alpha < \mu$, such that

$$\{a \in S\colon s_{\bigcup(a\cap\mu)} \cap a = A \cap a\} \in NS^+_{\kappa,\lambda} \quad \text{for all } A \subseteq \lambda.$$

Such a sequence will be called a $\diamondsuit_{\kappa\colon \mu\colon \lambda}(S)$-sequence.

Clearly $\diamondsuit_{\kappa:\,\mu:\,\lambda}(S)$ implies $\diamondsuit_{\kappa,\lambda}(S)$. Also notice that $\diamondsuit_{\kappa:\,\kappa:\,\kappa}(S)$ is equivalent to $\diamondsuit_\kappa(S \cap \kappa)$.

Just as $\diamondsuit_{\kappa,\lambda}(S)$ (see [14]), $\diamondsuit_{\kappa:\,\mu:\,\lambda}(S)$ can be reformulated in terms of partitions.

PROPOSITION 7.1: *The following are equivalent :*

(i) $\diamondsuit_{\kappa:\,\mu:\,\lambda}(S)$ *holds.*

(ii) *There exists* $H \in P(\mu)^\lambda$ *such that* $\{a \in S\colon \bigcup(a \cap \mu) \in \bigcap_{\alpha \in a} \varphi(\alpha)\} \in NS^+_{\kappa,\lambda}$ *for every* $\varphi \in \prod_{\alpha<\lambda}\{H(\alpha), \mu - H(\alpha)\}$.

Proof: (i)→(ii): Given $s_\alpha \in \bigcup_{\beta \in \mathrm{On}} P(\beta)$, $\alpha < \mu$, define $H \in P(\mu)^\lambda$ by letting $H(\beta) = \{\alpha < \mu\colon \beta \in s_\alpha\}$.

(ii)→(i): Given $H \in P(\mu)^\lambda$, set $s_\alpha = \{\beta < \lambda\colon \alpha \in H(\beta)\}$ for all $\alpha < \mu$. ∎

COROLLARY 7.2: *Assume* $\diamondsuit_{\kappa:\,\mu:\,\lambda}(S)$ *holds for some* $S \in NS^+_{\kappa,\lambda}$. *Then* $2^\mu \geq \lambda$ *and* $2^{<\kappa} \leq \mu$.

Proof: Let $H \in P(\mu)^\lambda$ be as in the statement of Proposition 7.1, and let $\nu \in (0, \kappa)$ be a cardinal. Then $\bigcap_{\alpha\in\nu} \varphi(\alpha) \neq 0$ for all $\varphi \in \prod_{\alpha\in\nu}\{H(\alpha), \mu - H(\alpha)\}$. Hence $2^\nu \leq \mu$. We leave it to the reader to verify that $2^\mu \geq \lambda$. ∎

Assume that λ is a strong limit cardinal of cofinality $< \kappa$, and let $S \in NS^+_{\kappa,\lambda}$. Then by Corollary 7.2, we have that for every $\mu \in [\kappa, \lambda]$, $\diamondsuit_{\kappa:\,\mu:\,\lambda}(S)$ does not hold. Thus our stronger version of diamond is trivially false in that case, so that our definition seems defective. However as we shall see below (Corollary 10.6), under the same assumptions $\diamondsuit_{\kappa,\lambda}(S)$ is trivially true, and thus not very meaningful either.

Clearly the definition of our principle makes it possible for a given sequence s_α, $\alpha < \mu$, to be a $\diamondsuit_{\kappa:\,\mu:\,\lambda}([\lambda]^{<\kappa})$-sequence for various κ's or λ's. The following two propositions illustrate that fact.

PROPOSITION 7.3: *Assume* s_α, $\alpha < \mu$, *is a* $\diamondsuit_{\kappa:\,\mu:\,\lambda}(S)$*-sequence, and let* $\nu \in [\mu, \lambda]$ *be a cardinal. Then* s_α, $\alpha < \mu$, *is a* $\diamondsuit_{\kappa:\,\mu:\,\nu}(\{a \cap \nu\colon a \in S\})$*-sequence.*

Proof: Left to the reader. ∎

PROPOSITION 7.4: *Let* $\rho \in (\omega, \kappa]$ *be a regular cardinal, and assume that* s_α, $\alpha < \mu$, *is a* $\diamondsuit_{\kappa:\,\mu:\,\lambda}(S)$*-sequence, where* $S \subseteq \{a \in [\lambda]^{<\kappa}\colon \mathrm{cof}(\bigcup(a \cap \mu)) < \rho\}$. *Let* T *be the set of all* $b \in [\lambda]^{<\rho}$ *such that there exists* $a \in S$ *with* $b \subseteq a$ *and* $\bigcup(b \cap \mu) = \bigcup(a \cap \mu)$. *Then* s_α, $\alpha < \mu$, *is a* $\diamondsuit_{\rho:\,\mu:\,\lambda}(T)$*-sequence.*

Proof: Left to the reader. ∎

Let us point out that the above result can be somewhat refined (see Proposition 11.4).

Throughout the remainder of this section, M will denote a fixed transitive model of ZFC, and λ will denote an uncountable cardinal of M.

Let us first recall (see [19]) the following fact. Suppose $(P,<)$ is, in M, an ω_1-closed notion of forcing, and let G be P-generic over M. Then $(NS^+_{\omega_1,\lambda})^M = (NS^+_{\omega_1,\lambda})^{M[G]} \cap M$. The following is also well-known.

LEMMA 7.5: *In M, assume that λ is regular, let $\kappa \in [\omega_1, \lambda]$ be a regular cardinal and let $(P,<)$ be a λ-closed notion of forcing. Let G be P-generic over M. Then $(NS^+_{\kappa,\lambda})^M = (NS^+_{\kappa,\lambda})^{M[G]} \cap M$.*

Proof: Let $p \in G$ and F be such that p forces that $F \in \lambda^{\lambda\times\lambda}$. Now working in M, construct $p_\alpha \in P$ and $f_\alpha \in \lambda^{\alpha\times\alpha}$, $\alpha < \lambda$, so that

(i) $p_0 \le p$;

(ii) $\beta < \alpha$ implies $p_\alpha \le p_\beta$;

(iii) p_α forces that $F|\alpha \times \alpha = f_\alpha$.

Finally set $f = \bigcup_{\alpha<\lambda} f_\alpha$. Given $S \in NS^+_{\kappa,\lambda}$, pick $a \in S \cap C_{f,\kappa}$ with $\cup a \notin a$. Then $p_{\cup a}$ forces that $a \in C_{F,\kappa}$. ∎

The following can be found in [15] (see Proposition 3.7 there).

PROPOSITION 7.6: *Assume that λ is regular in M, and add a Cohen subset of λ. Then in the extension, there is a sequence $s_\alpha \subseteq \lambda$, $\alpha < \lambda$, that is a $\diamondsuit_{\kappa:\,\lambda:\,\lambda}(S)$-sequence for every regular cardinal $\kappa \in (\omega, \lambda]$ and every $S \in NS^+_{\kappa,\lambda} \cap M$.*

If we restrict our attention to the case $\kappa = \omega_1$, then Proposition 7.6 can be generalized as follows.

PROPOSITION 7.7: *In M, let ρ, μ be uncountable cardinals such that ρ is regular, $\mathrm{cof}(\mu) \ge \rho$, $\mu \le \lambda$ and μ, $\lambda \notin (\rho, 2^{<\rho}]$. Let G be $\mathrm{Fn}(\lambda \times \mu, 2, \rho)$-generic over M. Then in $M[G]$, there is a sequence $s_\alpha \subseteq \lambda$, $\alpha < \mu$, that is a $\diamondsuit_{\omega_1:\,\mu:\,\nu}(S)$-sequence for every cardinal $\nu \in [\mu, \lambda]$ and every $S \in NS^+_{\omega_1,\nu} \cap M$.*

We omit the proof of Proposition 7.7, as it is very similar to the proof of the following.

PROPOSITION 7.8: *Let μ be a regular uncountable cardinal in M such that $\lambda \geq \mu$ and $\lambda \notin (\mu, 2^{<\mu}]$, and let G be $\mathrm{Fn}(\omega \times \mu, 2, \mu)$-generic over M. Then in $M[G]$, there are $s_\alpha \subseteq \omega$, $\alpha < \mu$, such that $\hat{a}[s_{\cup(a\cap\mu)}]$, $a \in [\lambda]^{<\omega_1}$, is a $\diamondsuit_{\omega_1,\lambda}(S)$-sequence for every $S \in NS^+_{\omega_1,\lambda} \cap M$.*

Proof: Set $s_\alpha = \{n \in \omega\colon (\cup G)(n, \alpha) = 1\}$ for all $\alpha < \mu$, and fix $S \in NS^+_{\omega_1,\lambda} \cap M$. Let $p \in G$ and B, F in M[G] be such that p forces that $B \in 2^\lambda$ and that $F \in \lambda^{\lambda\times\lambda}$. We will now work in M. Let p_γ, $\gamma < 2^{<\mu}$, be a one-to-one enumeration of the set of all $p' \leq p$. Let D denote the collection of all $x \in [\lambda \cdot 2^{<\mu}]^{<\omega_1}$ such that for every $\gamma \in x \cap 2^{<\mu}$, the following two conditions hold :

(0) Given $z\colon 2 \to x \cap \lambda$ and $\eta \in x \cap \lambda$, there are $\delta \in x \cap 2^{<\mu}$, $\xi \in x \cap \lambda$ and $m < 2$ such that $p_\delta \leq p_\gamma$, and p_δ forces that $F(z(0), z(1)) = \xi$ and that $B(\eta) = m$.

(1) Given $(n, \beta) \in \mathrm{dom}(p_\gamma)$, $\beta < \cup(x \cap \mu)$.

It is not difficult to see that D is a closed unbounded subset of $[\lambda \cdot 2^{<\mu}]^{<\omega_1}$. Now pick $x \in D$ such that $\omega \subseteq x$, $x \cap \lambda \in S$ and $\cup(x \cap \mu) \notin x$. Let z_n, $n \in \omega$, enumerate $(x \cap \lambda)^2$. For each $n \in \omega$, define $\gamma_n \in x \cap 2^{<\mu}$, $\xi_n \in x \cap \lambda$ and $m_n < 2$ so that

(0) $p_{\gamma_{n+1}} \leq p_{\gamma_n}$;

(1) p_{γ_n} forces that $F(z_n(0), z_n(1)) = \xi_n$ and that $B(\widehat{x \cap \lambda}(n)) = m_n$.

Finally set $q = (\bigcup_{n\in\omega} p_{\gamma_n}) \cup \{((n, \cup(x \cap \mu)), m_n)\colon n \in \omega\}$. Then q forces that $F[(x \cap \lambda) \times (x \cap \lambda)] \subseteq x \cap \lambda$ and that $\widehat{x \cap \lambda}[s_{\cup(x\cap\mu)}] = \{\alpha \in x \cap \lambda\colon B(\alpha) = 1\}$. ∎

Let us point out that if $\kappa > \omega_1$, then both Proposition 7.7 and Proposition 7.8 can be partially generalized using the methods of Section 14. To give an example, assume that in M, $2^{\aleph_1} = \aleph_2$, and add a Cohen subset of ω_2. Then in the extension, there are $s_\alpha \subseteq \omega_1$, $\alpha < \omega_2$, such that $\hat{a}[s_{\cup(a\cap\omega_2)}]$, $a \in [\omega_3]^{<\omega_2}$, is a $\diamondsuit_{\omega_2,\omega_3}(S)$-sequence for all $S \in (NS^+_{\omega_2,\omega_3})^M$ with $S \subseteq \{a\colon \mathrm{cof}(\cup a) \neq \omega_1$ or $\mathrm{cof}(\cup(a \cap \omega_2)) \neq \omega\}$.

8. $\diamondsuit^\rho_{\kappa,\lambda}$

Let $\rho > 1$ be a cardinal, and let $S \subseteq [\lambda]^{<\kappa}$. We say that $t_a \subseteq P(a)$, $a \in [\lambda]^{<\kappa}$, is a $\diamondsuit^\rho_{\kappa,\lambda}(S)$-sequence if for every $E \in [P(\lambda)]^{<\rho}$, the set $\{a \in S\colon t_a = \{A \cap a\colon A \in E\}\}$ is stationary in $[\lambda]^{<\kappa}$. The principle $\diamondsuit^\rho_{\kappa,\lambda}(S)$ asserts the existence of such a sequence.

We let $ND^{\rho}_{\kappa,\lambda}$ denote the set of all $S \subseteq [\lambda]^{<\kappa}$ such that $\diamondsuit^{\rho}_{\kappa,\lambda}(S)$ does not hold. Notice that $ND^{\rho}_{\kappa,\lambda} \subseteq ND^{\rho'}_{\kappa,\lambda}$ whenever $\rho' \geq \rho$. Also, $ND^{2}_{\kappa,\lambda} = ND_{\kappa,\lambda}$.

We observe that $ND^{\rho}_{\kappa,\lambda}$ is an ideal over $[\lambda]^{<\kappa}$ extending $NS_{\kappa,\lambda}$.

PROPOSITION 8.1: *Assume either that ρ is a successor, or else that $\mathrm{cof}(\rho) > \lambda$. Then $ND^{\rho}_{\kappa,\lambda}$ is a normal ideal.*

Proof: Let $Q_\alpha \in ND^{\rho}_{\kappa,\lambda}$, $\alpha < \lambda$. For each $\alpha < \lambda$, let W_α be the set of those $a \in [\lambda]^{<\kappa}$ such that $\alpha \in a$ and $a \in Q_\alpha - \bigcup_{\beta\in a\cap\alpha} Q_\beta$. Set $W = \bigcup_{\alpha<\lambda} W_\alpha$. Let a bijection $j\colon \lambda \times \lambda \to \lambda$ and a sequence $u_a \subseteq P(a)$, $a \in [\lambda]^{<\kappa}$, be given. Put $t_a = \{\{\beta \in a\colon j(\alpha,\beta) \in d\}\colon d \in u_a\}$ for all $a \in W_\alpha$. For each $\alpha < \lambda$, select $E_\alpha \in [P(\lambda)]^{<\rho}$ such that $\{a \in W_\alpha\colon t_a = \{A \cap a\colon A \in E_\alpha\}\} \in NS_{\kappa,\lambda}$. Put $\mu = \bigcup_{\alpha<\lambda} |E_\alpha|$, and for each $\alpha < \lambda$, choose $F_\alpha \in P(\lambda)^{\mu}$ with $\mathrm{ran}(F_\alpha) = E_\alpha$. Set $B_\gamma = \{j(\alpha,\beta)\colon \alpha < \lambda \text{ and } \beta \in F_\alpha(\gamma)\}$ for all $\gamma < \mu$, and let

$$S = \{a \in W\colon u_a = \{a \cap B_\gamma\colon \gamma < \mu\}\}.$$

Suppose $S \in NS^{+}_{\kappa,\lambda}$. Then one can find $\alpha \in \lambda$ and $T \in NS^{+}_{\kappa,\lambda}$ with $T \subseteq S \cap W_\alpha \cap \{a \in [\lambda]^{<\kappa}\colon j[a \times a] = a\}$. We have $t_a = \{F_\alpha(\delta) \cap a\colon \delta < \mu\}$ for all $a \in T$, a contradiction. ∎

An easy modification of the above argument shows that $ND^{\rho}_{\kappa,\lambda}$ is ν-complete for every cardinal $\nu \leq \kappa$ with $\mathrm{cof}(\rho) \geq \nu$.

This is a generalization of Proposition 1.6 of [15].

PROPOSITION 8.2: *Let $S \subseteq [\lambda]^{<\kappa}$ with $S \notin ND^{\rho}_{\kappa,\lambda}$. Then there are $t^{F}_{a} \subseteq P(a)$, $a \in [\lambda]^{<\kappa}$ and $F \in 2^{\lambda}$, such that $\{a \in S\colon \forall\alpha \in a\ t_a^{H(\alpha)} = \{q(\alpha) \cap a\colon q \in Q\}\} \in NS^{+}_{\kappa,\lambda}$ for all $Q \in [P(\lambda)^{\lambda}]^{<\rho}$ and one-to-one $H\colon \lambda \to 2^{\lambda}$.*

Proof: Let $s_a \subseteq P(a)$, $a \in [\lambda]^{<\kappa}$, be a $\diamondsuit^{\rho}_{\kappa,\lambda}(S)$-sequence. Select a bijection $v\colon \lambda \to \lambda \times \lambda \times \lambda \times \lambda$. For each $a \in [\lambda]^{<\kappa}$, choose $j_a\colon 2^a \to P(P(a))$ with the following property: in case

$$\{v[e]\colon e \in s_a\}$$
$$= \{\{(\alpha,\beta,\gamma,\delta) \in a \times a \times a \times a\colon h(\alpha)(\beta) = \gamma \text{ and } \delta \in w(\alpha)\}\colon w \in W\},$$

where $h\colon a \to 2^a$ is one-to-one and $W \subseteq P(a)^a$, then for every $\alpha \in a$, $j_a(h(\alpha)) = \{w(\alpha)\colon w \in W\}$.

Given $a \in [\lambda]^{<\kappa}$ and $F \in 2^{\lambda}$, set $t^{F}_{a} = j_a(F|a)$. Let $Q \in [P(\lambda)^{\lambda}]^{<\rho}$ and $H\colon \lambda \to 2^{\lambda}$ be given with H one-to-one. Put

$$K = \{\{(\alpha,\beta,\gamma,\delta) \in \lambda \times \lambda \times \lambda \times \lambda\colon H(\alpha)(\beta) = \gamma \text{ and } \delta \in q(\alpha)\}\colon q \in Q\}.$$

Let D denote the set of all $a \in [\lambda]^{<\kappa}$ such that $v[a] = a \times a \times a \times a$ and that the function $h\colon a \to 2^a$ defined by $h(\alpha) = H(\alpha)|a$ is one-to-one. Since D is closed and unbounded, the set T of all $a \in D \cap S$ with $s_a = \{a \cap v^{-1}[A]\colon A \in K\}$ is stationary in $[\lambda]^{<\kappa}$. It is readily checked that $t_a^{H(\alpha)} = \{q(\alpha) \cap a\colon q \in Q\}$ whenever $\alpha \in a \in T$. ∎

Corollary 8.3: *Given $S \in P([\lambda]^{<\kappa}) - ND^{\rho}_{\kappa,\lambda}$, there are pairwise disjoint $T_\alpha \notin ND^{\rho}_{\kappa,\lambda}$, $\alpha < \lambda^{<\kappa}$, with $\bigcup_{\alpha<\lambda^{<\kappa}} T_\alpha = S$.*

Proof: Let t^F_a, $a \in [\lambda]^{<\kappa}$ and $F \in 2^\lambda$, be as in the statement of Proposition 8.2. Let $G \in 2^\lambda$ be fixed. For each $d \in [\lambda]^{<\kappa}$, set $T_d = \{a \in S\colon t^G_a = \{d\}\}$. Pick a one-to-one $H\colon \lambda \to 2^\lambda$ with $H(0) = G$. Let $d \in [\lambda]^{<\kappa}$ and $E \in [P(\lambda)]^{<\rho}$ be given. Choose $Q \in [P(\lambda)^\lambda]^{<\rho}$ so that $\{q(0)\colon q \in Q\} = \{d\}$ and $\{q(1)\colon q \in Q\} = E$. Let Y denote the set of all $a \in S$ such that $2 \cup d \subseteq a$ and that for all $\alpha \in a$, $t_a^{H(\alpha)} = \{q(\alpha) \cap a\colon q \in Q\}$. Then Y is stationary, and $Y \subseteq \{a \in T_d\colon t_a^{H(1)} = \{A \cap a\colon A \in E\}\}$. ∎

We next show that the truth-value of $\diamondsuit^{\rho}_{\kappa,\lambda}(S)$ is the same for all small values of ρ, and for all large values of ρ.

Corollary 8.4: $ND_{\kappa,\lambda} = ND^{\kappa}_{\kappa,\lambda}$.

Proof: Let $S \in P([\lambda]^{<\kappa}) - ND_{\kappa,\lambda}$. By Proposition 8.2 there are $t^F_a \subseteq a$, $a \in [\lambda]^{<\kappa}$ and $F \in 2^\lambda$, such that $\{a \in S\colon \forall\alpha \in a\ t_a^{H(\alpha)} = q(\alpha) \cap a\} \in NS^+_{\kappa,\lambda}$ for all $q \in P(\lambda)^\lambda$ and one-to-one $H\colon \lambda \to 2^\lambda$. Choose such an H, and for each $a \in [\lambda]^{<\kappa}$, set $t_a = \{t_a^{H(\alpha)}\colon \alpha \in a\}$. Given $E \in \bigcup_{\alpha\in(0,\kappa)} P(\lambda)^\alpha$, let $q \in P(\lambda)^\lambda$ be such that $\operatorname{ran}(q) = \operatorname{ran}(E)$ and $q|\operatorname{dom}(E) = E$. Then $\{a \in S\colon \forall\alpha \in a\ t_a^{H(\alpha)} = q(\alpha) \cap a\} \cap \{a \in [\lambda]^{<\kappa}\colon \operatorname{dom}(E) \subseteq a\} \subseteq \{a \in S\colon t_a = \{E(\beta) \cap a\colon \beta \in \operatorname{dom}(E)\}\}$.
∎

Proposition 8.5: $ND^{(2^\lambda)^+}_{\kappa,\lambda} = ND^{(\lambda^{<\kappa})^+}_{\kappa,\lambda}$.

Proof: Let $S \in ND^{(2^\lambda)^+}_{\kappa,\lambda}$. Fix a sequence $t_a \subseteq P(a)$, $a \in [\lambda]^{<\kappa}$, and pick $E \subseteq P(\lambda)$ such that $\{a \in S\colon t_a = \{A \cap a\colon A \in E\}\} \in NS_{\kappa,\lambda}$. Set

$$W = \{b \in [\lambda]^{<\kappa}\colon t_b \subset \{A \cap b\colon A \in E\}\}.$$

For each $b \in W$, pick $A_b \in E$ with $A_b \cap b \notin t_b$. Now

$$\{a \in S\colon t_a = \{A_b \cap a\colon b \in W\}\} \in NS_{\kappa,\lambda}.$$

Thus $S \in ND_{\kappa,\lambda}^{([\lambda]^{<\kappa})^+}$. ∎

The following is easily verified.

PROPOSITION 8.6: *Assume* $[\lambda]^{<\kappa} \notin ND_{\kappa,\lambda}^{(2^\nu)^+}$, *where* ν *is an infinite cardinal* $< \kappa$. *Then* $2^{2^\nu} \leq \lambda^{<\kappa}$.

Thus if κ is not a strong limit, we have $P([\kappa]^{<\kappa}) = ND_{\kappa,\kappa}^{\kappa^+}$.

9. Games

Let $n < \omega$, and let $\mu_i \in (\kappa, \lambda]$, $i \leq n$, be a strictly decreasing sequence of regular cardinals. Set $\mu_{n+1} = \kappa$, and for each $i \leq n+1$, let $S_i \subseteq \{\alpha\colon \mathrm{cof}(\alpha) < \kappa\}$ be a stationary subset of μ_i. Fix $q \in n+1$, and assume that $S_q \subseteq \{\alpha\colon \mathrm{cof}(\alpha) = \omega\}$. Let X be the set of all $a \in [\lambda]^{<\kappa}$ such that $\bigcup(a \cap \mu_i) \in S_i$ for all $i \leq n+1$.

We assume the following :

(i) $2^{<\kappa} < \mu_q$ in case κ is either a limit or the successor of a cardinal with uncountable cofinality;

(ii) $\nu^{<\nu} < \mu_q$ in case $\kappa = \nu^+$ and $\mathrm{cof}(\nu) = \omega$.

Let $F \in \lambda^{\lambda\times\lambda}$ be given, and let T_0 and u be as in the proof of Proposition 3.7. Define $k\colon \mu_q^{<\omega} - \{0\} \to [\lambda]^{<\kappa}$ as follows :

(i) Assume κ is either a limit or the successor of a cardinal with uncountable cofinality. Then $k(\alpha_0, ..., \alpha_m) = e(F, u \cup \{\alpha_0, ..., \alpha_m\}, \kappa)$.

(ii) Assume that $\kappa = \nu^+$, where $\mathrm{cof}(\nu) = \omega$, and let cardinals $\nu_m < \nu$, $m < \omega$, be such that $\bigcup_{m<\omega} \nu_m = \nu$. Together with k we will define an auxiliary function $h\colon \mu_q^{<\omega} - \{0\} \to [\lambda]^{<\kappa}$:

(0) $h(\alpha_0) = \{\alpha_0\} \cup \widehat{u \cup \nu}[\nu_0]$;

(1) $k(\alpha_0) = h(\alpha_0) \cup F[h(\alpha_0) \times h(\alpha_0)]$;

(2) $h(\alpha_0, ..., \alpha_{m+1}) = k(\alpha_0, ..., \alpha_m) \cup \{\alpha_{m+1}\} \cup \widehat{u \cup \nu}[\nu_{m+1}]$;

(3) $k(\alpha_0, ..., \alpha_{m+1}) = h(\alpha_0, ..., \alpha_{m+1}) \cup F[h(\alpha_0, ..., \alpha_{m+1}) \times h(\alpha_0, ..., \alpha_{m+1})]$.

Now for every $\alpha < \kappa$, we let the infinite two-person game $G(\alpha)$ be played according to the following rules.

A move of Player I consists in selecting $\beta < \mu_q$ and $c \subset \alpha$. II answers each time by choosing $\gamma < \mu_q$. I (respectively II) thus constructs $\varphi \in \mu_q^\omega$ and $\chi \in P(\alpha)^\omega$ (resp. $\psi \in \mu_q^\omega$). I wins iff the following are satisfied :

(*) $\varphi(m+1) \geq \psi(m)$.

(**) There is a strictly increasing bijection $j_m\colon \chi(m) \to k(\varphi(0), ..., \varphi(m))$.

(***) $j_m \subseteq j_{m+1}$.

(****) $\bigcup(u \cap \mu_i) = \bigcup e(F, u \cup \operatorname{ran}(\varphi), \kappa) \cap \mu_i$ for all $i \leq n+1$ with $i \neq q$.

Let us observe that $e(F, u \cup \operatorname{ran}(\varphi), \kappa) = \bigcup_{m<\omega} k(\varphi(0), ..., \varphi(m))$.

Choose $\delta < \kappa$ and a stationary subset T_1 of T_0 such that for all $\alpha \in T_1$, o.t. $e(F, u \cup \operatorname{ran}(\tilde{\alpha}), \kappa) = \delta$.

LEMMA 9.1: I *has a winning strategy in* $G(\delta)$.

Proof: We will define a winning strategy σ for I in $G(\delta)$. Consider a run of the game where II plays $\psi \in \mu_q^\omega$. For each $\alpha \in T_1$, let $\overline{\alpha}$ be the increasing enumeration of $e(F, u \cup \operatorname{ran}(\tilde{\alpha}), \kappa)$. Inductively define X_m, β_m and c_m, $m < \omega$, as follows :

(i) X_0 is a stationary subset of T_1;

(ii) $\beta_0 \in \bigcap_{\alpha \in X_0} \operatorname{ran}(\tilde{\alpha})$;

(iii) $c_0 = \bar{\alpha}^{-1}[k(\beta_0)]$ for all $\alpha \in X_0$;

(iv) X_{m+1} is a stationary subset of X_m;

(v) $\beta_{m+1} \geq \psi(m)$;

(vi) $\beta_{m+1} \in \bigcap_{\alpha \in X_m} \operatorname{ran}(\tilde{\alpha})$;

(vii) $c_{m+1} = \bar{\alpha}^{-1}[k(\beta_0, ..., \beta_{m+1})]$ for all $\alpha \in X_{m+1}$.

Then set $\sigma(0) = (\beta_0, c_0)$ and $\sigma(\psi(0), ..., \psi(m)) = (\beta_{m+1}, c_{m+1})$.

Given $\alpha < \kappa$ and $\chi \in P(\alpha)^\omega$, we define a new game $G(\alpha, \chi)$ as follows. I (respectively II) constructs $\varphi \in \mu_q^\omega$ (resp. $\psi \in \mu_q^\omega$), and I wins iff (*), (**), (***) and (****) are verified.

LEMMA 9.2: *There are* $\delta < \kappa$ *and* $\xi \in P(\delta)^\omega$ *such that* I *has a winning strategy in* $G(\delta, \xi)$.

Proof: Let δ be as in the statement of Lemma 9.1, and let σ be a winning strategy for I in $G(\delta)$. We will define $\xi \in P(\delta)^\omega$ and a winning strategy σ' for I in $G(\delta, \xi)$. Let $\xi(0) = c_0$ and $\sigma'(0) = \beta_0$, where $(\beta_0, c_0) = \sigma(0)$. Given $\gamma_m < \mu_q$, $m \leq j$, set $B_c = \{\gamma < \mu_q \colon \exists \beta < \mu_q \ \sigma(\gamma_0, ..., \gamma_{j-1}, \gamma) = (\beta, c)\}$ for all $c \subset \alpha$. Then pick c with $|B_c| = \mu_q$, and put $\xi(j+1) = c$ and $\sigma'(\gamma_0, ..., \gamma_j) = \beta$, where $(\beta, c) = \sigma(\gamma_0, ..., \gamma_{j-1}, \bigcap(B_c - \gamma_j))$. ∎

LEMMA 9.3: *There are* $\alpha < \kappa$ *and* $\chi \in P(\alpha)^\omega$ *such that* $\bigcup_{m<\omega} \chi(m) = \alpha$ *and* I *has a winning strategy in* $G(\alpha, \chi)$.

Proof: Let δ, ξ be as in the statement of Lemma 9.2. Put $\alpha = \text{o.t.}(\bigcup_{m<\omega} \xi(m))$. Then letting $j \colon \alpha \to \bigcup_{m<\omega} \xi(m)$ be the increasing enumeration of $\bigcup_{m<\omega} \xi(m)$, set $\chi(m) = j^{-1}[\xi(m)]$ for all $m < \omega$. ∎

PROPOSITION 9.4: *Assume $\kappa = \omega_1$. Then $|C \cap X| = \lambda^{\aleph_0}$ for every $C \in NS^*_{\omega_1,\lambda}$.*

Proof: By Lemma 9.3, there are $\alpha < \omega_1$ and $\chi \in P(\alpha)^\omega$ such that $\bigcup_{m<\omega} \chi(m) = \alpha$ and I has a winning strategy in $G(\alpha, \chi)$. It is not difficult to find $Q \in \alpha^\omega$ such that I has a winning strategy σ in the game $G(\alpha, \chi, Q)$, which is defined as follows. I (respectively II) builds $\varphi \in \mu_q^\omega$ (resp. $\psi \in \mu_q^\omega$). I wins iff (*), (**), (***), (****) hold, and moreover $j_m(Q(m)) = \varphi(m)$. Let H: $\mu_q^{<\omega} \to \mu_q$ be such that $H(\gamma_0, ..., \gamma_m) = \cup(\mu_q \cap k(\sigma(0), ..., \sigma(\gamma_0, ..., \gamma_m)))$, and set $D = \{\gamma < \mu_q \colon H[\gamma^{<\omega}] \subseteq \gamma\}$. D clearly is a closed unbounded subset of μ_q. Select $\gamma \in D \cap S_q$.

Let $<$ be the total order on $\bigcup_{m<\omega}$ such that

(i) $f < g$ whenever $\mathrm{dom}(f) < \mathrm{dom}(g)$;

(ii) if f, $g \in 2^{m+1}$ and $f|m \neq g|m$, then $f < g$ iff $g|m < f|m$;

(iii) if $f \in 2^m$, then $f \cup \{(m, 0)\} < f \cup \{(m, 1)\}$ iff m is even.

Now by induction define β_f and γ_f, $f \in \bigcup_{m<\omega} 2^m$, as follows:

(a) $\beta_0 = \sigma(0)$;

(b) $\beta_f = \sigma(\gamma_{f|1}, \gamma_{f|2}, ..., \gamma_f)$ whenever $f \neq 0$;

(c) let f, g be such that $g < f$ and for every $h < f$, $h \leq g$. Then set $\gamma_f = \tilde{\gamma}(m)$ in case $g \in 2^m$ and $f \in 2^{m+1}$, and $\gamma_f = \beta_g + 1$ in case $\mathrm{dom}(f) = \mathrm{dom}(g)$.

For each $h \in 2^\omega$, set $x(h) = e(F, u \cup \{\beta_{h|m} \colon m \in \omega\}, \kappa)$. Clearly $x(h) \in C_{F,\kappa} \cap X$. Also $x(h) - x(h') \neq 0$ and $x(h') - x(h) \neq 0$ whenever $h \neq h'$. Finally observe that by Proposition 2.3, $\lambda^{\aleph_0} = 2^{\aleph_0} \cdot u(\omega_1, \lambda)$. ∎

Let $\rho > 1$ be a cardinal with $2^\eta < \mu_q$ for every cardinal $\eta < \rho$. Also let $E \in [P(\lambda)]^{<\rho}$ be given.

Given $\alpha < \kappa$, $\chi \in P(\alpha)^\omega$ and $\xi \in \prod_{m<\omega} P(\chi(m))^E$, we define a new game $G(\alpha, \chi, \xi)$ as follows. I (respectively II) builds $\varphi \in \mu_q^\omega$ (resp. $\psi \in \mu_q^\omega$). I wins iff (*), (**), (***), (****) hold and moreover for all $A \in E$, $j_m[\xi(m)(A)] = A \cap k(\varphi(0), ..., \varphi(m))$.

LEMMA 9.5: *There are $\alpha < \kappa$, $\chi \in P(\alpha)^\omega$ and $\xi \in \prod_{m<\omega} P(\chi(m))^E$ such that $\bigcup_{m<\omega} \chi(m) = \alpha$ and I has a winning strategy in $G(\alpha, \chi, \xi)$.*

Proof: By Lemma 9.3, one can find α, χ, σ such that $\bigcup_{m<\omega} \chi(m) = \alpha$ and σ is a winning strategy for I in $G(\alpha, \chi)$. We will define $\xi \in \prod_{m<\omega} P(\chi(m))^E$ and a winning strategy σ' for I in $G(\alpha, \chi, \xi)$. Let j_0: $\chi(0) \to k(\sigma(0))$ be onto and strictly increasing, and define $\xi(0)$ by letting $\xi(0)(A) = j_0^{-1}[A \cap k(\sigma(0))]$. Put $\sigma'(0) = \sigma(0)$. Given $\gamma_m < \mu_q$, $m \leq j$, let for every $\gamma < \mu_q$, J_γ: $\chi(j + 1) \to$

$k(\sigma(0),...,\sigma(\gamma_0,...,\gamma_{j-1},\gamma))$ be onto and strictly increasing. Set

$$B_d = \{\gamma < \mu_q : \forall A \in E\ J_\gamma^{-1}[A \cap k(\sigma(0),...,\sigma(\gamma_0,...,\gamma_{j-1},\gamma))] = d(A)\}$$

for all $d \in P(\chi(j+1))^E$. Then pick d with $|B_d| = \mu_q$, and put $\xi(j+1) = d$ and $\sigma'(\gamma_0,...,\gamma_j) = \sigma(\gamma_0,...,\gamma_{j-1},\bigcap(B_d - \gamma_j))$. ∎

Let us now make one further assumption on κ. We assume that $2^\nu \leq \mu_q$ in case $\kappa = \nu^+$ and $\mathrm{cof}(\nu) = \omega$.

PROPOSITION 9.6: $X \notin ND^\rho_{\kappa,\lambda}$.

Proof: First choose pairwise disjoint $T_Y \subset S_q$, $Y \in \bigcup_{\alpha<\kappa}[P(\alpha)]^{<\rho}$, so that each T_Y is a stationary subset of μ_q. Given $a \in X$, let $j{:}: \mathrm{o.t.}a \to a$ be the increasing enumeration of a. If $\bigcup(a \cap \mu_q) \in T_Y$ for some Y, set $t_a = \{j[y \cap \mathrm{o.t.}a]: y \in Y\}$; otherwise put $t_a = 0$. Now let $F: \lambda \times \lambda \to \lambda$ and $E \in [P(\lambda)]^{<\rho}$ be given. Let α, χ, ξ be as in the statement of Lemma 9.5, and let σ be a winning strategy for I in $G(\alpha,\chi,\xi)$. Set $Y = \{\bigcup_{m<\omega}\xi(m)(A): A \in E\}$. Let $H: \mu_q^{<\omega} \to \mu_q$ be such that $H(\gamma_0,...,\gamma_m) = \bigcup(\mu_q \cap k(\sigma(0),...,\sigma(\gamma_0,...,\gamma_m)))$, and set

$$D = \{\gamma < \mu_q : H[\gamma^{<\omega}] \subseteq \gamma\}.$$

Select $\gamma \in D \cap T_Y$, and put $a = \bigcup_{m<\omega} k(\sigma(0),...,\sigma(\tilde{\gamma}(0),...,\tilde{\gamma}(m)))$. Then $\mathrm{o.t.}a = \alpha$ and $\bigcup(a \cap \mu_q) = \gamma$. Moreover $a \in X \cap C_{F,\kappa}$ and $t_a = \{A \cap a: A \in E\}$. ∎

The following corollary states the special case of our result when $\kappa = \omega_1$ and $\lambda = 2^{\aleph_0}$.

COROLLARY 9.7: *Assume $2^{\aleph_0}$ is a regular cardinal with $2^{\aleph_0} > \omega_1$, and let $S \subseteq \{\alpha: \mathrm{cof}(\alpha) = \omega\}$ be a stationary subset of $2^{\aleph_0}$. Then*

$$\{a \in [2^{\aleph_0}]^{<\omega_1}: \bigcup a \in S\} \notin ND_{\omega_1,2^{\aleph_0}}.$$

A simple trick can be used to cover more cases. For example, assuming $2^{<\aleph_\omega} = \aleph_{\omega+\omega}$, we obtain that $[\aleph_{\omega+\omega}]^{<\aleph_n} \notin ND^{\aleph_\omega}_{\aleph_n,\aleph_{\omega+\omega}}$ for all $n \in [1,\omega)$. This is how we do it.

PROPOSITION 9.8: *Let $\rho \in (\kappa,\lambda]$ be a cardinal such that $2^\eta < \lambda$ for every cardinal $\eta < \rho$. Then $[\lambda]^{<\kappa} \notin ND^\rho_{\kappa,\lambda}$.*

Proof: The result is immediate from Proposition 9.6 in case ρ is a successor. Thus assume ρ is a limit, and let $\rho_\alpha < \rho$, $\alpha < \mathrm{cof}(\rho)$, be a sequence of cardinals with $\bigcup_{\alpha<\mathrm{cof}(\rho)}\rho_\alpha=\rho$. Select pairwise disjoint stationary subsets S_α of $\kappa\cdot\mathrm{cof}(\rho)$, $\alpha<\mathrm{cof}(\rho)$, with each $S_\alpha\subseteq\{\beta\colon \mathrm{cof}(\beta)<\kappa\}$. For each $\alpha<\mathrm{cof}(\rho)$, let X_α be the set of all $a\in[\lambda]^{<\kappa}$ such that $\mathrm{cof}(\bigcup(a\cap(2^{\kappa\cdot\rho_\alpha})^+))=\omega$ and $\bigcup(a\cap\kappa\cdot\mathrm{cof}(\rho))\in S_\alpha$. By Proposition 9.6 $X_\alpha\notin ND^{\rho_\alpha}_{\kappa,\lambda}$ for all α with $(2^{\kappa\cdot\rho_\alpha})^+\neq\mathrm{cof}(\rho)$. It is easily seen that $\bigcup_{\alpha<\mathrm{cof}(\rho)}X_\alpha\notin ND^{\rho}_{\kappa,\lambda}$. ∎

10. $\diamondsuit_{\kappa,\nu,\lambda}$

Let ν be a cardinal with $\kappa\le\nu\le\lambda$. Given $S\subseteq[\lambda]^{<\kappa}$, the principle $\diamondsuit_{\kappa,\nu,\lambda}(S)$ asserts the existence of a sequence $s_a\subseteq\bigcup a$, $a\in[\lambda]^{<\kappa}$, such that for all $A\subseteq\lambda$, the set $\{a\in S\colon s_a=A\cap(a\cup\bigcup(a\cap\nu))\}$ is stationary in $[\lambda]^{<\kappa}$.

Let $ND_{\kappa,\nu,\lambda}$ be the set of all $S\subseteq[\lambda]^{<\kappa}$ such that $\diamondsuit_{\kappa,\nu,\lambda}(S)$ does not hold.

$ND_{\kappa,\nu,\lambda}$ is easily shown to be an ideal. We have that $ND_{\kappa,\nu,\lambda}\subseteq ND_{\kappa,\nu',\lambda}$ whenever $\nu'\ge\nu$. Also, $ND_{\kappa,\kappa,\lambda}=ND_{\kappa,\lambda}$.

The following is easily checked.

Proposition 10.1: (i) *If $\mathrm{cof}(\nu)\ge\kappa$ and $[\lambda]^{<\kappa}\notin ND_{\kappa,\nu,\lambda}$, then $2^{<\nu}\le\lambda^{<\kappa}$.*
(ii) *If $\mathrm{cof}(\nu)<\kappa$ and $[\lambda]^{<\kappa}\notin ND_{\kappa,\nu,\lambda}$, then $2^{\nu}\le\lambda^{<\kappa}$.*

Proposition 10.2: (i) *If $\mathrm{cof}(\lambda)<\kappa$, then $ND^{\lambda^+}_{\kappa,\lambda}\subseteq ND_{\kappa,\lambda,\lambda}$.*
(ii) *If $\mathrm{cof}(\lambda)\ge\kappa$, then $ND^{\lambda}_{\kappa,\lambda}\subseteq ND_{\kappa,\lambda,\lambda}$.*

Proof: Set $\rho=\lambda$ in case $\mathrm{cof}(\lambda)\ge\kappa$, and $\rho=\lambda^+$ otherwise. Fix $S\in ND^+_{\kappa,\lambda,\lambda}$. We will show that there are $t_a\subseteq P(\bigcup a)$, $a\in S$, such that

$$\{a\in S\colon t_a=\{A\cap\bigcup a\colon A\in E\}\}\in NS^+_{\kappa,\lambda}\quad\text{for all } E\in[P(\lambda)]^{<\rho}.$$

We will first define a one-to-one function $j\colon\lambda\times\lambda\to\lambda$. If λ is regular or $\mathrm{cof}(\lambda)<\kappa$, then let j be arbitrary. Now assume $\kappa\le\mathrm{cof}(\lambda)<\lambda$. Let μ_γ, $\gamma<\mathrm{cof}(\lambda)$, be a strictly increasing sequence of infinite cardinals such that $\lambda=\bigcup_{\gamma<\mathrm{cof}(\lambda)}\mu_\gamma$, and that for every limit ordinal $\gamma\in(0,\mathrm{cof}(\lambda))$, $\mu_\gamma=\bigcup_{\delta<\gamma}\mu_\delta$. Let $j_0\colon\mu_0\times\mu_0\to\mu_0$ be one-to-one, and for every $\gamma<\mathrm{cof}(\lambda)$, let $j_{\gamma+1}\colon\mu_{\gamma+1}\times\mu_{\gamma+1}\to\mu_{\gamma+1}-\mu_\gamma$ be one-to-one. Then set $j(\alpha,\beta)=j_\delta(\alpha,\beta)$, where δ is least with $\{\alpha,\beta\}\subset\mu_\delta$. Let $s_a\subseteq\bigcup a$, $a\in S$, be such that $\{a\in S\colon s_a=A\cap\bigcup a\}\in NS^+_{\kappa,\lambda}$ for all $A\subseteq\lambda$. Given $a\in S$, put $d^\alpha_a=\{\beta\in\bigcup a\colon j(\alpha,\beta)\in s_a\}$ for all $\alpha<\lambda$. Then set $t_a=\{0\}$ in case $\bigcup_{\alpha<\lambda}d^\alpha_a=0$, and $t_a=\{d^\alpha_a\colon\bigcup_{\beta\ge\alpha}d^\beta_a\neq 0\}$ otherwise. Now let $\mu<\rho$ and $E_\alpha\subseteq\lambda$, $\alpha<\mu$, be given such that μ is an infinite cardinal and for every α, $|\{\beta<\mu\colon E_\alpha=E_\beta\}|=\mu$. Put $A=\bigcup_{\alpha<\mu}\{j(\alpha,\beta)\colon\beta\in E_\alpha\}$. Suppose $a\in S$

is such that $\bigcup a \notin a$, $s_a = A \cap \bigcup a$ and for every $\beta \in a$, $j[\mu \times \beta] \subseteq \bigcup a$. Then $t_a = \{E_\alpha \cap a: \alpha < \mu\}$. ∎

PROPOSITION 10.3: *$ND_{\kappa,\nu,\lambda} = ND_{\kappa,\lambda}$ whenever $2^{<\nu} \leq \lambda$.*

Proof: Choose bijections $j: 2 \times \lambda \to \lambda$ and $h: 2^{<\nu} \to \bigcup_{\alpha<\nu} P(\alpha)$. Assume $S \subseteq [\lambda]^{<\kappa}$ and $s_a \subseteq a$, $a \in [\lambda]^{<\kappa}$, are such that $\{a \in S: s_a = A \cap a\} \in NS^+_{\kappa,\lambda}$ for all $A \subseteq \lambda$. For each a, set

$$u_a = \{\beta < \lambda: j(0,\beta) \in s_a\}, \quad v_a = \bigcup h[\{\gamma < 2^{<\nu}: j(1,\gamma) \in s_a\}],$$

and finally $t_a = (v_a \cap \bigcup(a \cap \nu)) \cup u_a$. Now let $B \subseteq \lambda$ be given, and define $f: \nu \to 2^{<\nu}$ by letting $f(\alpha) = h^{-1}(B \cap \alpha)$. Set

$$A = \{j(0,\beta): \beta \in B\} \cup \{j(1, f(\alpha)): \alpha < \nu\}.$$

Suppose $a \in S$ is such that $A \cap a = s_a$, $2 \subseteq a$, $j[2 \times a] = a$ and $f[a \cap \nu] \subseteq a$. Then $t_a = B \cap (a \cup \bigcup(a \cap \nu))$. ∎

COROLLARY 10.4: *Assuming the Generalized Continuum Hypothesis, $ND_{\kappa,\lambda} = ND_{\kappa,\lambda,\lambda} = ND^{\lambda^{<\kappa}}_{\kappa,\lambda}$.*

Proof: By Proposition 10.3 and Proposition 10.2. ∎

PROPOSITION 10.5: *Let $\rho > \lambda$ be a cardinal, and let $S \subseteq [\lambda]^{<\kappa}$ be such that $|D \cap S| = 2^{<\rho}$ for every closed unbounded subset D of $[\lambda]^{<\kappa}$. Then there exist $t_a \subseteq P(\lambda)$, $a \in [\lambda]^{<\kappa}$, such that $\{a \in S: t_a = E\} \in NS^+_{\kappa,\lambda}$ for all $E \in [P(\lambda)]^{<\rho}$.*

Proof: Fix a bijection $j: \lambda^{<\kappa} \to [P(\lambda)]^{<\rho} \times \lambda^{\lambda\times\lambda}$. Now by induction on $\alpha < \lambda^{<\kappa}$, define $a_\alpha \in S$ and $t_{a_\alpha} \subseteq P(\lambda)$ so that

(0) $\beta < \alpha$ implies $a_\beta \neq a_\alpha$;
(1) $a_\alpha \cap \kappa \in \kappa$;
(2) if $j(\alpha) = (E, F)$, then $F[a_\alpha \times a_\alpha] \subseteq a_\alpha$ and $t_{a_\alpha} = E$. ∎

Notice that given $S \subseteq [\lambda]^{<\kappa}$, S splits into 2^λ many pairwise disjoint stationary sets iff there are $t_a \subseteq \lambda$, $a \in [\lambda]^{<\kappa}$, such that $\{a \in S: t_a = A\} \in NS^+_{\kappa,\lambda}$ for all $A \subseteq \lambda$.

COROLLARY 10.6: *Assume λ is a strong limit with $\mathrm{cof}(\lambda) < \kappa$. Then $ND_{\kappa,\lambda} = NS_{\kappa,\lambda}$.*

Proof: By Corollary 2.4 and Proposition 10.5. ∎

COROLLARY 10.7: *Assume that* $\lambda \in [\omega_2, 2^{\aleph_0})$ *and that* $2^{<2^{\aleph_0}} = 2^{\aleph_0}$. *Let* $b \subseteq [\omega_1, \lambda]$ *be a finite family of regular cardinals, and for each* $\mu \in b$, *let* $S_\mu \in NS^+_\mu$ *with* $S_\mu \subseteq \{\alpha\colon \mathrm{cof}(\alpha) = \omega\}$. *Then* $\{a \in [\lambda]^{<\omega_1}\colon \forall\mu \in b \cup (a \cap \mu) \in S_\mu\} \in ND^{(2^\lambda)^+}_{\omega_1,\lambda} - ND^{2^\lambda}_{\omega_1,\lambda}$.

Proof: By Proposition 9.4, Proposition 10.5 and Proposition 8.6. ∎

11. From one diamond sequence to another

PROPOSITION 11.1: *Let* $\rho > 1$ *and* $\nu > \lambda$ *be cardinals, and let* $S \in P([\lambda]^{<\kappa}) - ND^\rho_{\kappa,\lambda}$. *Further let* $T \in NS^+_{\lambda^+,\nu}$, *and* $g_a \in \lambda^a$ *and* $D_a \in NS^*_{\kappa,a}$, $a \in T$, *be such that each* g_a *is a bijection, and* $g_a[y] = g_d[y]$ *whenever* $a, d \in T$ *and* $y \in D_a \cap D_d$. *Then* $\{y \in \bigcup_{a\in T} D_a\colon y \cap \lambda \in S\} \notin ND^\rho_{\kappa,\nu}$.

Proof: Let $t_b \subseteq P(b)$, $b \in [\lambda]^{<\kappa}$, be a $\diamondsuit^\rho_{\kappa,\lambda}(S)$-sequence. Given $a \in T$ and $b \in S$ such that $\lambda \subseteq a$ and $g_a^{-1}[b] \in D_a$, set $u_{g_a^{-1}[b]} = \{g_a^{-1}[x]\colon x \in t_b\}$. Now fix $E \in [P(\nu)]^{<\rho}$ and $F_0\colon \nu \times \nu \to \nu$. Select $a \in T \cap C_{F_0,\lambda^+}$ such that $\lambda \subseteq a$ and $\cup a \notin a$. It easily follows from Proposition 1.5 of [16] that there exists $F_1\colon a \times a \to a$ such that $\{d \in [a]^{<\kappa}\colon d \cap \kappa \in \kappa \text{ and } F_1[d \times d] \subseteq d\} \subseteq \{0\} \cup D_a$. For each $i < 2$, define $G_i\colon \lambda \times \lambda \to \lambda$ by letting $G_i(\alpha,\beta) = g_a(F_i(g_a^{-1}(\alpha), g_a^{-1}(\beta)))$. Then select $b \in S$ so that

(i) $b \in C_{G_0,\kappa} \cap C_{G_1,\kappa}$;
(ii) $g_a^{-1}[b] \cap \lambda = b$;
(iii) $t_b = \{b \cap g_a[A \cap a]\colon A \in E\}$.

It is easily checked that $g_a^{-1}[b] \in D_a \cap D_{F_0,\kappa}$, and that

$$u_{g_a^{-1}[b]} = \{A \cap g_a^{-1}[b]\colon A \in E\}.$$ ∎

Notice that the results of Section 5 give the following. Let $\nu > \lambda$ be such that either $\nu < \lambda^{+\kappa}$, or $u(\lambda^+, \mu) \le \nu$ for every cardinal $\mu \in [\lambda^+, \nu)$, and $\mathrm{cof}(\nu) \notin [\kappa, \lambda]$. Then one can find $T \in NS^+_{\lambda^+,\nu}$ such that there exist pairwise disjoint $D_a \in NS^*_{\kappa,a}$, $a \in T$. On the other hand, the existence of such a T clearly implies that $s(\lambda^+, \nu) \le \nu^{<\kappa}$.

PROPOSITION 11.2: *Let* $\nu > \lambda$ *be a cardinal such that either* $\nu = \lambda^{+\kappa}$, *or* $\nu = \bigcup_{\mu\in(\lambda,\nu)} u(\lambda^+, \mu)$ *and* $\mathrm{cof}(\nu) \in [\kappa, \lambda]$. *Then one can find* $T \in NS^+_{\lambda^+,\nu}$, *and*

$g_a \in \lambda^a$ and $D_a \in NS^*_{\kappa,a}$, $a \in T$, such that each g_a is a bijection, and $g_a[y] = g_d[y]$ whenever a, $d \in T$ and $y \in D_a \cap D_d$.

Proof: Select a bijection j: $\mathrm{cof}(\nu) \times \lambda \to \lambda$. Also choose cardinals $\nu_\alpha < \nu$, $\alpha < \mathrm{cof}(\nu)$, such that

(i) $\nu_0 = 0$ and $\nu_1 = \lambda^+$;

(ii) $\nu_\alpha < \nu_{\alpha+1}$;

(iii) $\nu_\alpha = \bigcup_{\beta<\alpha} \nu_\beta$ whenever α is an infinite limit ordinal;

(iv) $\nu = \bigcup_{\alpha<\mathrm{cof}(\nu)} \nu_\alpha$.

Let Z be the set of all $a \in [\nu]^{<\lambda^+}$ such that $|a \cap [\nu_\alpha, \nu_{\alpha+1})| = \lambda$ for every $\alpha < \mathrm{cof}(\nu)$. We are going to define $T \subseteq Z$, and r: $\mathrm{cof}(\nu) \times T \to [\nu]^{<\kappa}$ such that $r(\alpha, a) \subseteq a$. Suppose that has been done, and let $a \in T$ be given. We let D_a be the set of all $y \in [a]^{<\kappa}$ such that $r(\alpha, a) \subseteq y$ for every $\alpha < \mathrm{cof}(\nu)$ with $y \cap [\nu_\alpha, \nu_{\alpha+1}) \neq 0$. We define g_a: $a \to \lambda$ so that $g_a(\xi) = j(\alpha, (a \cap [\nu_\alpha, \nu_{\alpha+1}))^{-1}(\xi))$ whenever $\xi \in a \cap [\nu_\alpha, \nu_{\alpha+1})$. Let us now define T and r. Let us first consider the case when $\nu = \lambda^{+\kappa}$. We let T be the set of all $a \in C_{\kappa,\lambda}$ such that $|a \cap [\lambda^{+\gamma}, \lambda^{+(\gamma+1)})| = \lambda$ and $\mathrm{cof}(\cup(a \cap \lambda^{+(\gamma+1)})) < \kappa$ for every $\gamma \in \kappa$. Clearly $T \in NS^+_{\kappa,\lambda}$. For every $a \in T$, define R_a: $\kappa \to [a]^{<\kappa}$ so that $\cup R_a(\gamma) = \cup(a \cap [\lambda^{+\gamma}, \lambda^{+(\gamma+1)}))$. Then set $r(\alpha, a) = \bigcup_{\gamma \in q_\alpha} R_a(\gamma)$, where $q_\alpha = \{\gamma < \kappa \colon \lambda^{+\gamma} < \nu_{\alpha+1}\}$.

Now for the other case. Select h: $\nu \to [\nu]^{<\lambda^+}$ with $\mathrm{ran}(h) \in UB(\lambda^+, \nu)$, and fix a regular cardinal $\mu \in [\omega, \kappa)$. Let T be the set of all $a \in Z$ such that $\bigcup_{\beta \in a} h(\beta) \subseteq a$, and there exists R: $\mathrm{cof}(\nu) \to [a]^\mu$ with $a \cap \nu_{\alpha+1} = \bigcup_{\beta \in R(\alpha)} h(\beta) \cap \nu_{\alpha+1}$. The definition of r should be clear. It remains to show that $T \in NS^+_{\lambda^+,\nu}$. Thus let F: $\nu \times \nu \to \nu$ be given such that $\bigcup_{\beta \in a} h(\beta) \subseteq a$ whenever $\lambda \subseteq a$ and $F[a \times a] \subseteq a$. Define a_γ, $\gamma < \mu$, and δ^α_γ, $\gamma < \mu$ and $\alpha < \mathrm{cof}(\nu)$, so that:

(0) $a_0 = \bigcup_{\alpha<\mathrm{cof}(\nu)} \{\nu_\alpha + \zeta \colon \zeta < \lambda\}$;

(1) $(a_\gamma \cup F[a_\gamma \times a_\gamma]) \cap \nu_{\alpha+1} \subseteq h(\delta^\alpha_\gamma)$;

(2) $a_{\gamma+1} = \{\delta^\alpha_\gamma \colon \alpha < \mathrm{cof}(\nu)\} \cup \bigcup_{\alpha<\mathrm{cof}(\nu)} h(\delta^\alpha_\gamma)$;

(3) $a_\gamma = \bigcup_{\zeta<\gamma} a_\zeta$ whenever γ is an infinite limit ordinal.

Then set $a = \bigcup_{\gamma<\mu} a_\gamma$. We have that $F[a \times a] \subseteq a$, $\lambda \subseteq a$ and $a \in T$, as desired. ∎

A modification of the proof of Proposition 11.1 yields the following.

PROPOSITION 11.3: *Assume λ is a strong limit with $\mathrm{cof}(\lambda) < \kappa$. Let $S \in NS^+_{\kappa,\lambda}$, and let $T \in NS^+_{\lambda^+}$ with $T \subseteq \{\alpha \colon \mathrm{cof}(\alpha) < \kappa\}$. Then*

$$\{y \in [\lambda^+]^{<\kappa} \colon y \cap \lambda \in S \text{ and } \cup y \in T\} \in ND^+_{\kappa,\lambda^+,\lambda^+}.$$

Proof: By Corollary 10.6 and Proposition 10.3, there are $s_b \subseteq \lambda$, $b \in [\lambda]^{<\kappa}$, such that $\{b \in S\colon s_b = A\} \in NS^+_{\kappa,\lambda}$ for all $A \subseteq \lambda$. Given $\alpha \in T$ and $b \in S$ such that $\lambda \subseteq a$ and $\mathrm{ran}(\tilde{\alpha}) \subseteq \hat{\alpha}[b]$, set $u_{\hat{\alpha}[b]} = \hat{\alpha}[s_b]$. Then proceed as in the proof of Proposition 11.1. ∎

PROPOSITION 11.4: *Let $\rho > 1$ be a cardinal, and let $\mu \in (\kappa,\lambda]$ be a regular cardinal. Let $T \in P([\lambda]^{<\mu}) - ND^{\rho}_{\mu,\lambda}$, and let $w_a \in NS^+_{\kappa,a}$, $a \in T$, be a pairwise disjoint family. Then $\bigcup_{a\in T} w_a \notin ND^{\rho}_{\kappa,\lambda}$.*

Proof: Let $t_a \subseteq P(a)$, $a \in [\lambda]^{<\mu}$, be a $\diamondsuit^{1+\rho}_{\mu,\lambda}(T)$-sequence. Select bijections $j\colon \lambda\times\lambda\times\lambda \to \lambda$ and $g\colon 2\times\lambda \to \lambda$. Given $A \subseteq \lambda$ and $f\colon A\times A \to A$, set $G_f = \{(\alpha,\beta,f(\alpha,\beta))\colon \alpha,\beta \in A\}$ and $B_f = \{g(1,j(x))\colon x \in G_f\}$. Let S consist of all $a \in T$ such that $\kappa \subseteq a$ and $|\{f \in a^{a\times a}\colon B_f \in t_a\}| = 1$. Given $a \in S$, let f_a be the unique $f \in a^{a\times a}$ with $B_f \in t_a$, and let v_a be the set of all $z \in w_a$ such that $z\cap\kappa \in \kappa$ and $f_a[z\times z] \subseteq z$. Now pick $z_a \in v_a$, $a \in S$. We will show that $\{z_a\colon a \in S\} \notin ND^{\rho}_{\kappa,\lambda}$. For each $a \in S$, set $y_a = \{A \in t_a\colon A \cap g[\{1\}\times\lambda] = 0\}$ and $u_{z_a} = \{\{\alpha \in z_a\colon g(0,\alpha) \in A\}\colon A \in y_a\}$. Now select $F\colon \lambda\times\lambda \to \lambda$ and $E \in [P(\lambda)]^{<\rho}$. Put $K = \{\{g(0,\alpha)\colon \alpha \in A\}\colon A \in E\} \cup \{B_F\}$, and choose $a \in T$ such that $t_a = \{B \cap a\colon B \in K\}$, $F[a\times a] \subseteq a$, $g[2\times a] = a$, $j[a\times a\times a] = a$ and $\kappa \subseteq a$. Clearly, $a \in S$ and $f_a = F|a\times a$. Thus $z_a \in C_{F,\kappa}$, and moreover $u_{z_a} = \{A \cap z_a\colon A \in E\}$. ∎

Let us observe the following. Suppose there exist pairwise disjoint $w_a \in NS^+_{\kappa,a}$, $a \in T$, where $T \in ND^+_{\mu,\lambda}$. Then $\lambda^{<\mu} = \lambda^{<\kappa}$.

The following is the analogue of Proposition 11.4 for the principle $\diamondsuit_{\kappa,\lambda,\lambda}$.

PROPOSITION 11.5: *Let $\mu \in (\kappa,\lambda]$ be a regular cardinal, let $T \in ND^+_{\mu,\lambda,\lambda}$, and let $w_a \in NS^+_{\kappa,a}$, $a \in T$, be a pairwise disjoint family. Then $\bigcup_{a\in T} w_a \in ND^+_{\kappa,\lambda,\lambda}$.*

Proof: Left to the reader. ∎

12. $\diamondsuit^{*\rho}_{\kappa,\lambda}$

Let $\rho > 1$ be a cardinal. Given $S \subseteq [\lambda]^{<\kappa}$, the principle $\diamondsuit^{*\rho}_{\kappa,\lambda}(S)$ asserts the existence of a sequence $w_a \in [P(P(a))]^{\leq|a|}$, $a \in [\lambda]^{<\kappa}$, such that for all $E \in [P(\lambda)]^{<\rho}$, $\{a \in S\colon \{A \cap a\colon A \in E\} \notin w_a\} \in NS_{\kappa,\lambda}$.

We let $D^{*\rho}_{\kappa,\lambda}$ be the set of all $S \subseteq [\lambda]^{<\kappa}$ such that $\diamondsuit^{*\rho}_{\kappa,\lambda}(S)$ holds.

Notice that $D^{*\rho'}_{\kappa,\lambda} \subseteq D^{*\rho}_{\kappa,\lambda}$ whenever $\rho' \geq \rho$. Also, $D^{*2}_{\kappa,\lambda} = D^{*}_{\kappa,\lambda}$.

PROPOSITION 12.1: *$D^{*\rho}_{\kappa,\lambda}$ is a normal ideal over $[\lambda]^{<\kappa}$ extending $NS_{\kappa,\lambda}$.*

Proof: Assume $S_\alpha \in D^{*\rho}_{\kappa,\lambda}$, $\alpha < \lambda$. For each α, let $w^\alpha_a \in [P(P(a))]^{\leq |a|}$, $a \in [\lambda]^{<\kappa}$, be such that for all $E \in [P(\lambda)]^{<\rho}$, $\{a \in S\colon \{A \cap a\colon A \in E\} \notin w^\alpha_a\} \in NS_{\kappa,\lambda}$. Set $w_a = w^\alpha_a$ whenever α, a are such that $\alpha \in a$ and $a \in S_\alpha - \bigcup_{\beta \in a \cap \alpha} S_\beta$. Now fix $E \in [P(\lambda)]^{<\rho}$. For each $\alpha < \lambda$, pick a closed unbounded subset C_α of $[\lambda]^{<\kappa}$ such that $C_\alpha \cap S_\alpha \subseteq \{a\colon \{A \cap a\colon A \in E\} \in w^\alpha_a\}$. Suppose $a \neq 0$ is such that $a \in \triangle_{\gamma<\lambda} C_\gamma$ and $a \in \bigcup_{\alpha \in a} S_\alpha$. Then $\{A \cap a\colon A \in E\} \in w_a$. ∎

An easy modification of the proof of Proposition 8.2 yields the following.

PROPOSITION 12.2: *Let $S \in D^{*\rho}_{\kappa,\lambda}$. Then there are $w^F_a \in [P(P(a))]^{\leq |a|}$, $a \in [\lambda]^{<\kappa}$ and $F \in 2^\lambda$, such that $\{a \in S\colon \forall \alpha \in a \ \{q(\alpha) \cap a\colon q \in Q\} \notin w^{H(\alpha)}_a\} \in NS_{\kappa,\lambda}$ for all $Q \in [P(\lambda)^\lambda]^{<\rho}$ and one-to-one $H\colon \lambda \to 2^\lambda$.*

PROPOSITION 12.3: *$D^*_{\kappa,\lambda} = D^{*\kappa}_{\kappa,\lambda}$.*

Proof: Let $S \in D^*_{\kappa,\lambda}$, and let $r_a \in P(a)^a$, $a \in [\lambda]^{<\kappa}$, be such that

$$\{a \in S\colon A \cap a \notin \operatorname{ran}(r_a)\} \in NS_{\kappa,\lambda} \quad \text{for all } A \subseteq \lambda.$$

Fix a bijection $j\colon \kappa \times \lambda \to \lambda$, and denote by D the set of all $a \in [\lambda]^{<\kappa}$ with $j[(a \cap \kappa) \times a] = a$. For each a, define $w_a\colon a \to P(P(a))$ by letting $w_a(\alpha) = \{\{\beta \in a\colon j(\delta,\beta) \in r_a(\alpha)\}\colon \delta \in a \cap \kappa\}$. Let $G \in P(\lambda)^\nu$ be given, where ν is a cardinal with $0 < \nu < \kappa$. Let A denote the set of all $j(\delta,\beta)$, where either $\delta < \nu$ and $\beta \in G(\delta)$, or else $\nu \leq \delta < \kappa$ and $\beta \in G(0)$. Suppose α, a are such that $\alpha \in a \in S \cap D$, $\nu \subseteq a$ and $r_a(\alpha) = A \cap a$. Then $w_a(\alpha) = \{G(\delta) \cap a\colon \delta < \nu\}$. ∎

PROPOSITION 12.4: *$D^{*(2^\lambda)^+}_{\kappa,\lambda} = D^{*(\lambda^{<\kappa})^+}_{\kappa,\lambda}$.*

Proof: Suppose $S \subseteq [\lambda]^{<\kappa}$ is such that $S \notin D^{*(2^\lambda)^+}_{\kappa,\lambda}$, and let a sequence $w^\alpha_a \subseteq P(a)$, $\alpha \in a \in [\lambda]^{<\kappa}$, be given. Select $E \subseteq P(\lambda)$ such that

$$\{a \in S\colon \forall \alpha \in a \{A \cap a\colon A \in E\} \neq w^\alpha_a\} \in NS^+_{\kappa,\lambda}.$$

Let P be the collection of all (α, a) such that $\alpha \in a \in [\lambda]^{<\kappa}$ and $w^\alpha_a \subset \{A \cap a\colon A \in E\}$. Given $(\alpha, a) \in P$, choose $A_{a,\alpha} \in E$ with $A_{a,\alpha} \cap a \notin w^\alpha_a$. Then

$$\{a \in S\colon \{A_{a,\alpha} \cap a\colon (\alpha, a) \in P\} \neq w^\alpha_a\} \in NS^+_{\kappa,\lambda}. \quad ∎$$

PROPOSITION 12.5: *Let ν be a limit cardinal with $\mathrm{cof}(\nu) \leq \lambda$. Then $D^{*\nu}_{\kappa,\lambda} = \bigcap_{\rho<\nu} D^{*\rho}_{\kappa,\lambda}$.*

Proof: Choose cardinals $\mu_\alpha \in (0,\nu)$, $\alpha < \mathrm{cof}(\nu)$, such that $\nu = \bigcup_{\alpha<\mathrm{cof}(\nu)} \mu_\alpha$, and fix $S \in \bigcap_{\alpha<\mathrm{cof}(\nu)} D^{*\mu_\alpha}_{\kappa,\lambda}$. For each $\alpha < \mathrm{cof}(\nu)$, let $w^\alpha_a \in [P(P(a))]^{\leq|a|}$, $a \in [\lambda]^{<\kappa}$, be such that $\{a \in S\colon \{A \cap a\colon A \in E\} \notin w^\alpha_a\} \in NS_{\kappa,\lambda}$ for all $E \in [P(\lambda)]^{<\mu_\alpha}$. Set $w_a = \bigcup_{\alpha\in a} w^\alpha_a$ for every $a \in [\lambda]^{<\kappa}$. It is easy to see that $\{a \in S\colon \{A \cap a\colon A \in E\} \notin w_a\} \in NS_{\kappa,\lambda}$ for all $E \in [P(\lambda)]^{<\nu}$. Hence $S \in D^{*\nu}_{\kappa,\lambda}$. ∎

PROPOSITION 12.6: *Assume either that ρ is a successor, or else that $\mathrm{cof}(\rho) > \lambda$. Then $D^{*\rho}_{\kappa,\lambda} \cap ND^{\rho}_{\kappa,\lambda} = NS_{\kappa,\lambda}$.*

Proof: Let $S \subseteq [\lambda]^{<\kappa}$ and $w_a\colon a \to P(P(a))$, $a \in [\lambda]^{<\kappa}$, be given such that $\{a \in S\colon \{A\cap a\colon A \in E\} \in \mathrm{ran}(w_a)\} \in NS^+_{\kappa,\lambda}$ for all $E \in [P(\lambda)]^{<\rho}$. Fix a bijection $j\colon \lambda \times \lambda \to \lambda$, and set $C = \{a \in [\lambda]^{<\kappa}\colon j[a \times a] = a\}$. Given $\alpha < \lambda$ and $a \in [\lambda]^{<\kappa}$ with $\alpha \in a$, set $t^\alpha_a = \{\{\beta \in a\colon j(\alpha,\beta) \in d\}\colon d \in w_a(\alpha)\}$. Suppose that for every $\alpha < \lambda$, there exist $E_\alpha \in [P(\lambda)]^{<\rho}$ and a closed unbounded subset D_α of $[\lambda]^{<\kappa}$ such that $t^\alpha_a \neq \{A \cap a\colon A \in E_\alpha\}$ whenever $\alpha \in a \in D_\alpha$. Put $\mu = \bigcup_{\alpha<\lambda} |E_\alpha|$, and for each $\alpha < \lambda$, choose $F_\alpha \in P(\lambda)^\mu$ with $\mathrm{ran}(F_\alpha) = E_\alpha$. Then define $F \in P(\lambda)^\mu$ by setting $F(\gamma) = \{j(\alpha,\beta)\colon \beta \in F_\alpha(\gamma)\}$. It is easy to find $T \subseteq S \cap C \cap \triangle_{\alpha<\lambda} D_\alpha$ with $T \in NS^+_{\kappa,\lambda}$ and $\alpha < \lambda$ such that $\{F(\gamma) \cap a\colon \gamma < \mu\} = w_a(\alpha)$ whenever $\alpha \in a \in T$, which yields a contradiction. ∎

This is a version for two cardinals of a well-known result of Gregory and Shelah (see Theorem 32 of [18]).

PROPOSITION 12.7: *Assume $2^{<\lambda} = \lambda$, and let S be the set of all $a \in [\lambda]^{<\kappa}$ such that $\mathrm{cof}(\bigcup a) \neq \mathrm{cof}(|a|)$ and that for every infinite cardinal $\mu < |a|$, $\mu^{\mathrm{cof}(\bigcup a)} \leq |a|$. Then $S \in D^*_{\kappa,\lambda}$.*

Proof: First choose $h\colon \lambda \to \bigcup_{\gamma<\lambda} P(\gamma)$ such that $|h^{-1}(b)| = \lambda$ for all $b \in \bigcup_{\gamma<\lambda} P(\gamma)$. For every $a \in S$, set $w_a = \{\bigcup_{\delta\in d} h(\hat{a}(\delta)) \cap \bigcup a\colon d \in \bigcup_{\beta<|a|} [\beta]^{\mathrm{cof}(\bigcup a)}\}$. Now fix $A \subseteq \lambda$. Select $g\colon \lambda \to \lambda$ such that $h(g(\alpha)) = A \cap \alpha$ and $g(\alpha) \geq \alpha$. Let D be the set of all $a \in [\lambda]^{<\kappa}$ such that $g[a] \subseteq a$ and $\bigcup a \notin a$. Given $a \in D \cap S$, pick $b \subseteq a$ such that $\bigcup b = \bigcup a$ and o.t. $b = \mathrm{cof}(\bigcup a)$. Then choose $d \subseteq \hat{a}^{-1}[g[b]]$ with o.t. $d = \mathrm{cof}(\bigcup a)$. Clearly $A \cap \bigcup a = \bigcup_{\delta\in d} h(\hat{a}(\delta))$. ∎

The following is now immediate.

COROLLARY 12.8: *Assuming the Generalized Continuum Hypothesis,*

$$\{a \in [\lambda]^{<\kappa}\colon \operatorname{cof}(\bigcup a) \neq \operatorname{cof}(|a|)\} \in D^*_{\kappa,\lambda}.$$

Corollary 12.8 can be used to show the following.

PROPOSITION 12.9: *Assume that the Generalized Continuum Hypothesis holds, and that $\lambda > \kappa$. Let $n \in \omega$, and let $\mu_i \in [\kappa, \lambda]$, $i \leq n+1$, be a strictly decreasing sequence of regular cardinals. For each $i \leq n+1$, let $S_i \in NS^+_{\mu_i}$ with $S_i \subseteq \{\alpha\colon \operatorname{cof}(\alpha) < \kappa\}$. Then $S \in ND^+_{\kappa,\lambda}$, where*

$$S = \{a \in [\lambda]^{<\kappa}\colon \forall i \leq n+1 \bigcup (a \cap \mu_i) \in S_i\}.$$

Proof: Wlog assume that $\mu_{n+1} = \kappa$ and $\mu_n = \kappa^+$. Select $h\colon \mu_0 \to [\mu_0]^{<\kappa^+}$ with $\operatorname{ran}(h) \in UB(\kappa^+, \mu_0)$, and define $\varphi\colon [\mu_0]^{<\kappa} \to [\mu_0]^{<\kappa^+}$ by letting $\varphi(a) = \bigcup_{\alpha \in a} h(\alpha)$. Let Y be the set of all $a \in [\mu_0]^{<\kappa}$ such that

(0) $\forall i \leq n \bigcup (a \cap \mu_i) \in S_i$;

(1) $a \cup \kappa \subseteq \varphi(a)$;

(2) $\forall i \leq n \bigcup (a \cap \mu_i) = \bigcup(\varphi(a) \cap \mu_i)$.

Then set $T = \{b \in \varphi[Y]\colon \bigcup_{\beta \in b} h(\beta) \subseteq b\}$. By Proposition 3.6, Proposition 5.5 and Corollary 12.8, $T \in ND^+_{\kappa^+,\mu_0}$. Define $\psi\colon T \to Y$ so that $\varphi(\psi(b)) = b$. For each $b \in T$, let w_b be the set of all $d \in [b]^{<\kappa}$ such that $\psi(b) \subseteq d$ and $d \cap \kappa \in S_{n+1}$. By Proposition 11.4, $\bigcup_{b \in T} w_b \in ND^+_{\kappa,\mu_0}$. Finally set

$$X = \{a \in [\lambda]^{<\kappa}\colon a \cap \mu_0 \in \bigcup_{b \in T} w_b\}.$$

By Proposition 11.1 and Proposition 11.2, $X \in ND^+_{\kappa,\lambda}$. It remains to observe that $X \subseteq S$. ∎

Note the sharp contrast with the results of [22], which deals with the one cardinal situation.

If one keeps in mind Proposition 5.6, the following can be seen as another generalization of the result of Gregory [8].

PROPOSITION 12.10: *Let $\rho \geq \kappa$ be a cardinal with $2^{<\rho} \leq \lambda$, let $h\colon \lambda \to [\lambda]^{<\kappa}$, and let $\nu < \kappa$ be an infinite cardinal. Then $\{a \in U^h_\nu\colon |a|^\nu = |a|\} \in D^{*\rho}_{\kappa,\lambda}$.*

Proof: For every $\alpha \in \lambda$, let $m_\alpha\colon \bigcup_{0<\mu<\rho}(2^{h(\alpha)})^\mu \to 2^{<\rho}$ be one-to-one. Given $a \in U^h_\nu$, pick $d_a \in [a]^\nu$ with $a = \bigcup_{\alpha \in d_a} h(\alpha)$. Then let w_a consist of all functions x such that

(i) $\mathrm{dom}(x)$ is a cardinal;
(ii) $0 < \mathrm{dom}(x) < \rho$;
(iii) $\mathrm{ran}(x) \subseteq 2^a$;
(iv) if $\alpha \in d_a$ and if $y\colon \mathrm{dom}(x) \to 2^{h(\alpha)}$ is given by $y(\beta) = x(\beta)|h(\alpha)$, then $m_\alpha(y) \in a$.

Fix $H\colon \mu \to 2^\lambda$, where $\mu \in (0, \rho)$ is a cardinal. Let D be the set of all $a \in [\lambda]^{<\kappa}$ such that if $\alpha \in a$ and if $y\colon \mu \to 2^{h(\alpha)}$ is given by $y(\beta) = H(\beta)|h(\alpha)$, then $m_\alpha(y) \in a$. Given $a \in D \cap U^h_\nu$, we have $x \in w_a$, where $x\colon \mu \to 2^a$ is given by $x(\beta) = H(\beta)|a$. ∎

13. $\diamondsuit^*_{\kappa,\nu,\lambda}$

Let ν be a cardinal with $\kappa \leq \nu \leq \lambda$. Given $S \subseteq [\lambda]^{<\kappa}$, $\diamondsuit^*_{\kappa,\nu,\lambda}(S)$ asserts the existence of a sequence $w_a \in [P(\cup a)]^{\leq|a|}$, $a \in [\lambda]^{<\kappa}$, such that for all $A \subseteq \lambda$, $\{a \in S\colon A \cap (a \cup \cup(a \cap \nu)) \notin w_a\} \in NS_{\kappa,\lambda}$.

We let $D^*_{\kappa,\nu,\lambda}$ be the set of all $S \subseteq [\lambda]^{<\kappa}$ such that $\diamondsuit^*_{\kappa,\nu,\lambda}(S)$ holds.

Notice that $D^*_{\kappa,\nu',\lambda} \subseteq D^*_{\kappa,\nu,\lambda}$ whenever $\nu' \geq \nu$. Also, $D^*_{\kappa,\kappa,\lambda} = D^*_{\kappa,\lambda}$.

The following two propositions are respectively proved as Proposition 10.2 and Proposition 10.3.

PROPOSITION 13.1: (i) *Assume* $\mathrm{cof}(\lambda) < \kappa$. *Then* $D^*_{\kappa,\lambda,\lambda} \subseteq D^{*\lambda^+}_{\kappa,\lambda}$.

(ii) *Assume* $\mathrm{cof}(\lambda) \geq \kappa$. *Then* $D^*_{\kappa,\lambda,\lambda} \subseteq D^{*\lambda}_{\kappa,\lambda}$.

PROPOSITION 13.2: $D^*_{\kappa,\nu,\lambda} = D^*_{\kappa,\lambda}$ *whenever* $2^{<\nu} \leq \lambda$.

COROLLARY 13.3: *Assuming the Generalized Continuum Hypothesis,* $D^*_{\kappa,\lambda} = D^*_{\kappa,\lambda,\lambda} = D^{*\lambda^{<\kappa}}_{\kappa,\lambda}$.

Proof: By Proposition 13.1 and Proposition 13.2. ∎

PROPOSITION 13.4: *Let* ν, ρ *be cardinals such that* $\nu < \kappa$, $\rho > \lambda$ *and* $2^{<\rho} = \lambda^\nu$. *Then there exist* $w_a \in [P(P(\lambda))]^{\leq|a|}$, $a \in [\lambda]^{<\kappa}$, *such that for all* $E \in [P(\lambda)]^{<\rho}$, $\{a \in [\lambda]^{<\kappa}\colon |a|^\nu = |a|$ *and* $E \notin w_a\} \in NS_{\kappa,\lambda}$.

Proof: Let E_d, $d \in [\lambda]^\nu$, be an enumeration of $[P(\lambda)]^{<\rho}$. Now for each $a \in [\lambda]^{<\kappa}$ with $|a|^\nu = |a|$, set $w_a = \{E_d\colon d \in [a]^\nu\}$. ∎

Let us observe the following. Given $S \subseteq [\lambda]^{<\kappa}$, let $Q(S)$ mean that there exist $w_a \in [P(\lambda)]^{\leq|a|}$, $a \in [\lambda]^{<\kappa}$, such that for all $A \subseteq \lambda$, $\{a \in S\colon A \notin w_a\} \in NS_{\kappa,\lambda}$. Then $Q(S)$ can be seen as a multidimensional version of the splitting property for S. Let us for instance consider the case $\kappa = \omega_1$ (which is easily generalized). Then $Q(S)$ holds iff there are $T_{\alpha,n} \subseteq S$, $\alpha < 2^\lambda$ and $n \in \omega$, such that (i) $\alpha \neq \beta$ implies $T_{\alpha,n} \cap T_{\beta,n} = 0$; and (ii) $S - \bigcup_{n\in\omega} T_{\alpha,n} \in NS_{\kappa,\lambda}$.

We will make use of the following fact. Let ν, ρ be infinite cardinals such that $\nu^{<\rho} = \nu$ and $\nu^\rho > \nu$. Then ρ is regular.

The following should be compared with Proposition 12.7.

PROPOSITION 13.5: *Let $\nu < \kappa$ and $\rho > \lambda$ be cardinals such that $2^{<\rho} = \lambda^\nu$. Assume either that $\nu = \mathrm{cof}(\lambda)$ and $\lambda^\nu = \lambda^+$, or that $\lambda^{<\nu} = \lambda$. Then there exist $w_a \in [P(P(\lambda))]^{\leq |a|}$, $a \in [\lambda]^{<\kappa}$, such that for all $E \in [P(\lambda)]^{<\rho}$,*

$$\{a \in [\lambda]^{<\kappa}\colon \mathrm{cof}(|a|) \neq \nu, \bigcup_{\eta<|a|} \eta^\nu \leq |a| \text{ and } E \notin w_a\} \in NS_{\kappa,\lambda}.$$

Proof: By Theorems 1.1.3 and 1.1.4 of [23], there exist $b_\alpha \in [\lambda]^\nu$, $\alpha < \lambda^\nu$, with the property that $|b_\alpha \cap b_\beta| < \nu$ whenever α, β are distinct members of λ^ν. Let $g\colon \bigcup_{\alpha<\lambda^\nu} [b_\alpha]^\nu \to [P(\lambda)]^{<\rho}$ be onto and such that for every α, g is constant on $[b_\alpha]^\nu$. For every $a \in [\lambda]^{<\kappa}$, set $w_a = \{g(\hat{a}[e])\colon e \subseteq |a| \text{ and o.t. } e = \nu\}$. Let $E \in [P(\lambda)]^{<\rho}$ be given, and let $\alpha < \lambda^\nu$ be such that $g(b_\alpha) = E$. Suppose now that $a \in [\lambda]^{<\kappa}$ is such that $b_\alpha \subset a$, $\mathrm{cof}(|a|) \neq \nu$ and for every cardinal $\eta < |a|$, $\eta^\nu \leq |a|$. Then $|w_a| \leq |a|$, and letting $j\colon \text{o.t.}(\hat{a}^{-1}[b_\alpha]) \to \hat{a}^{-1}[b_\alpha]$ be the increasing enumeration of $\hat{a}^{-1}[b_\alpha]$, $g(\hat{a}[j[\nu]]) = E$. ∎

The following is now easily derived.

COROLLARY 13.6: *Assume the Singular Cardinals Hypothesis. If $2^{\mathrm{cof}(\lambda)} < \kappa$ and $2^{<\lambda} \leq \lambda^+$, then $\{a \in [\lambda]^{<\kappa}\colon \mathrm{cof}(\bigcup a) \neq \mathrm{cof}(|a|\} \in D^*_{\kappa,\lambda,\lambda}$.*

14. Forcing the failure of $\diamondsuit^*_{\kappa,\lambda}$

Throughout this section, M will denote a fixed transitive model of ZFC, κ an uncountable regular cardinal of M, and λ an uncountable cardinal of M.

We first show that adding one Cohen subset of ω_1 is enough to destroy all $\diamondsuit^*_{\omega_1,\lambda}(S)$-sequences of the ground model.

PROPOSITION 14.1: *Let P be an ω_1-closed p.o. in M. Let G be P-generic over M, and assume that $M[G]$ contains an order type ω_1 subset of λ that is not in M, and that λ remains a cardinal in $M[G]$. In M, let $S \in NS^+_{\omega_1,\lambda}$ and*

$w_a \in [P(a)]^{<2^{\aleph_0}}, a \in [\lambda]^{<\omega_1}$, be given. Then in $M[G]$, there is $A \subseteq \lambda$ with $\{a \in S\colon A \cap a \notin w_a\} \in NS^+_{\omega_1,\lambda}$.

Proof: Let $p \in G$ and r, F in $M[G]$ be such that p forces that $F \in \lambda^{\lambda\times\lambda}$ and that r is a strictly increasing function from ω_1 to λ with $r \notin M$. Let us now work in M. Set $\theta = |\{p'\colon p' \leq p\}|$, and let p_γ, $\gamma < \theta$, be a one-to-one enumeration of the set of all $p' \leq p$. Define $k\colon \theta \to \omega_1$ by letting $k(\gamma)$ be the least $\zeta \in \omega_1$ such that for each $\sigma \in \lambda$, p_γ does not force that $r(\zeta) = \sigma$. For each $\gamma < \theta$, let d_γ be the set of those $\sigma \in \lambda$ such that p_γ forces that $r(\zeta) = \sigma$ for some $\zeta < k(\gamma)$. We let D be the collection of those $x \in [\theta \cdot \lambda]^{<\omega_1}$ such that for every $\gamma \in x \cap \theta$, the following two conditions are satisfied :

(0) $d_\gamma \subseteq x$;

(1) given $\alpha, \beta \in x \cap \lambda$, there exist $\delta_i \in x \cap \theta$ and ξ_i, $\pi_i \in x \cap \lambda$, $i < 1$, such that $\pi_0 \neq \pi_1$ and for each $i < 1$, $p_{\delta_i} \leq p_\gamma$ and p_{δ_i} forces that $F(\alpha, \beta) = \xi_i$ and that $r(k(\gamma)) = \pi_i$.

Then D is a closed unbounded subset of $[\theta \cdot \lambda]^{<\omega_1}$. Now pick $x \in D$ such that $\omega \subseteq x$ and $x \cap \lambda \in S$. Let z_n, $n < \omega$, enumerate $(x \cap \lambda)^2$. For each $u \in 2^{<\omega}$, define $\gamma_u \in x \cap \theta$ and ξ_u, $\pi_u \in x \cap \lambda$ so that

(0) $u \subseteq v$ implies $p_{\gamma_v} \leq p_{\gamma_u}$;

(1) p_{γ_u} forces that $F(z_n(0), z_n(1)) = \xi_u$, where $n = \mathrm{dom}(u)$;

(2) setting $v_i = u \cup \{(\mathrm{dom}(u), i)\}$ for $i < 1$, $\pi_{v_0} \neq \pi_{v_1}$ and for each i, $p_{\gamma_{v_i}}$ forces that $r(k(\gamma_u)) = \pi_{v_i}$.

Select $f \in 2^\omega$ such that for all $b \in w_{x\cap\lambda}$, $\bigcup_{u\subset f} d_{\gamma_u} \neq b \cap \bigcup_{u\subset f} \pi_u$. Pick $q \in P$ such that $q \leq p_{\gamma_u}$ for all $u \subset f$. Then q forces that $F[(x \cap \lambda) \times (x \cap \lambda)] \subseteq x \cap \lambda$ and that $\mathrm{ran}(r) \cap x \notin w_{x\cap\lambda}$. ∎

We then keep adding Cohen subsets of ω_1 until all potential $\diamondsuit^*_{\omega_1,\lambda}(S)$-sequences are destroyed.

COROLLARY 14.2: *In M, assume that $\lambda \notin \omega_1 \cup (\omega_1, 2^{\aleph_0}]$ and let ν be a cardinal with $\nu > \lambda^{\aleph_0}$. Let G be $\mathrm{Fn}(\nu \times \omega_1, 2, \omega_1)$-generic over M. Then in $M[G]$, $D^*_{\omega_1,\lambda} = NS_{\omega_1,\lambda}$.*

Proof: For each $A \subseteq \nu$, let G_A consist of all $q \in G$ such that $\mathrm{dom}(q) \subseteq A \times \omega_1$. In $M[G]$, let $W \in \prod_{a\in[\lambda]^{<\omega_1}} [P(a)]^{<\omega_1}$ and $S \in NS^+_{\omega_1,\lambda}$ be given. Then there exists $A \subsetneq \nu$ such that $|A| \leq \lambda^{\aleph_0}$ and S, $W \in M[G_A]$. Now observe that $M[G] = M[G_A][G_{\nu-A}]$, and apply Proposition 14.1. ∎

The obvious drawback of the method is that $2^{\aleph_1}$ is large in the generic extension. More precisely if we set in M $\theta = \nu^{(2^{\aleph_0})}$, then in $M[G]$, $\nu \leq 2^{\aleph_1} \leq \theta$.

Let us now turn to the case when $\kappa > \omega_1$. Proposition 14.1 is unfortunately not so easily generalized.

PROPOSITION 14.3: *In M, let $\mu \in [\kappa, \lambda]$ and $\rho \in [\omega, \kappa)$ be regular cardinals such that $\nu^{<\rho} < \kappa$ for all cardinals $\nu \in [\omega, \kappa)$, and $2^{<\mu} = \mu$. Further let in M, $w_a \in [P(a)]^{\leq |a|}$, $a \in [\lambda]^{<\kappa}$, $h\colon \lambda \to [\lambda]^{<\kappa}$ and $S \in NS^+_{\kappa,\lambda}$ with $S \subseteq \{a \in \cup^h_\rho \colon |a|^\rho > |a|\}$. In case $\rho > \omega$, assume that $\mu < \kappa^{+\omega}$, and that $\mathrm{cof}(\cup(a \cap \tau)) = \rho$ for all $a \in S$ and all cardinals $\tau \in [\kappa, \mu]$. Let G be $\mathrm{Fn}(\mu, \kappa, \mu)$-generic over M. Then in $M[G]$, $\{a \in S\colon A \cap a \notin w_a\} \in NS^+_{\kappa,\lambda}$ for some $A \subseteq \mu$.*

Proof: Let $p \in G$ and F be such that p forces that $F \in \lambda^{\lambda\times\lambda}$. Let us now work in M. Let Q be the set of all $q \in \mathrm{Fn}(\mu, \kappa, \mu)$ such that $q \leq p$ and $\mathrm{dom}(q) \in \mu$. Select bijections $\varphi\colon \mu \to Q$ and $j\colon \mu \times \kappa \to \mu$. Let D be the set of all $a \in [\lambda]^{<\kappa}$ such that

(0) $\mathrm{dom}(\varphi(\gamma)) \in a$ for all $\gamma \in a \cap \mu$.

(1) Given α, $\beta \in a$, $\gamma \in a \cap \mu$ and $\delta \in a \cap \kappa$, there is $\eta \in a \cap \mu$ such that $\varphi(\eta) < \varphi(\gamma)$, $\varphi(\eta)(\mathrm{dom}(\varphi(\gamma))) = \delta$ and $\varphi(\eta)$ forces that $F[h(\alpha)\times h(\beta)] \subseteq a$.

(2) $j[(a \cap \mu) \times (a \cap \kappa)] = a \cap \mu$.

(3) $\gamma \in a$ whenever there are $\alpha \in a \cap \kappa$ and $b \in [\alpha]^{<\rho}$ such that $\varphi(\gamma) = \bigcup_{\beta\in b} \varphi(\beta)$.

(4) Let $n \in (0,\omega)$, let $\alpha_i \in a \cap \mu$, $i \leq n$, be such that $|\alpha_{j+1}| > \alpha_j$ for all $j < n$, and that $\alpha_1 > \kappa > \alpha_0$, and let $b \in [\alpha_0]^{<\rho}$. If $\gamma \in \mu$ is such that $\varphi(\gamma) = \bigcup_{\beta\in b} \varphi(\hat{\alpha_n}(\cdots(\hat{\alpha_1}(\beta))\cdots))$, then $\gamma \in a$.

Now pick $a \in D \cap C_{\kappa,\lambda} \cap S$ such that $|a| = |a \cap \kappa|$. Select $d \in [a]^\rho$ so that $a = \bigcup_{\alpha\in d} h(\alpha)$, and let z_α, $\alpha < \rho$, be an enumeration of d^2. Put $R = \bigcup_{\alpha\leq\rho}(a \cap \kappa)^\alpha$. We define $q_f \in Q$, $f \in R$, so that the following hold :

(i) $f' \subseteq f$ implies $q_f < q_{f'}$;

(ii) if $\mathrm{dom}(f) = \alpha + 1$, then $\varphi^{-1}(q_f) \in a$, $q_f(\mathrm{dom}(q_{f|\alpha})) = f(\alpha)$ and q_f forces that $F[h(z_\alpha(0)) \times h(z_\alpha(1))] \subseteq a$;

(iii) if $\mathrm{dom}(f)$ is an infinite limit ordinal, then $q_f = \bigcup_{f'\subset f} q_{f'}$.

Finally pick $f\colon \rho \to a \cap \kappa$ so that $j[q_f] \cap a \notin \{b \cap j[\mathrm{dom}(q_f) \times \kappa]\colon b \in w_a\}$. Then q_f forces that $F[a \times a] \subseteq a$ and that $j[\cup G] \cap a \notin w_a$. ∎

COROLLARY 14.4: *In M, let $n \in \omega$ be such that $2^{<\kappa^{+n}} = \kappa^{+n}$, and let ρ be an infinite regular cardinal such that $\tau^{<\rho} < \kappa$ for all cardinals $\tau \in [\omega, \kappa)$. Assume* *that κ is either a limit, or else the successor of a cardinal ν with $\nu^\rho > \nu$. Let G be $\mathrm{Fn}(\kappa^{+(n+1)} \times \kappa^{+n}, 2, \kappa^{+n})$-generic over M. Then in $M[G]$, $S \notin D^*_{\kappa,\kappa^{+n}}$ for all $S \in NS^+_{\kappa,\kappa^{+n}}$ with $S \subseteq \{a\colon \forall i \leq n \ \mathrm{cof}(\bigcup(a \cap \kappa^{+i})) = \rho\}$.*

Proof: Set $G_B = \{p \in G\colon \mathrm{dom}(p) \subseteq B \times \kappa^{+n}\}$ for all $B \subseteq \kappa^{+(n+1)}$. In $M[G]$, let $S \in NS^+_{\kappa,\kappa^{+n}}$ with $S \subseteq \{a\colon \forall i \leq n \ \mathrm{cof}(\bigcup(a \cap \kappa^{+i})) = \rho\}$, and let $w_a \in [P(a)]^{\leq |a|}$, $a \in [\kappa^{+n}]^{<\kappa}$. Then there exists $\beta < \kappa^{+(n+1)}$ such that both S and the sequence w_a, $a \in [\kappa^{+n}]^{<\kappa}$, lie in $M[G_\beta]$. Now by Proposition 5.6 and Proposition 14.3, there are in $M[G_\beta][G_{\{\beta\}}]$, $A \subseteq \kappa^{+n}$ and $T \in NS^+_{\kappa,\kappa^{+n}}$ such that $T = \{a \in S\colon A \cap a \notin w_a\}$. It remains to observe that by Lemma 7.5, T remains stationary in $M[G]$. ∎

An easy modification of the proof of Proposition 14.3 yields the following.

PROPOSITION 14.5: *In M, assume that κ is strongly inaccessible, let $\mu \in [\kappa, \lambda]$ be a regular cardinal with $\mu < \kappa^{+\omega}$ and $2^{<\mu} = \mu$, and let $w_a \in [P(a)]^{\leq |a|}$, $a \in [\lambda]^{<\kappa}$, and $h\colon \lambda \to [\lambda]^{<\kappa}$. Further let in M $S \in NS^+_{\kappa,\lambda}$ be such that for each $a \in S$, there is some cardinal ρ such that $a \in \bigcup^h_\rho$, $|a|^\rho > |a|$ and for every cardinal $\tau \in [\kappa, \mu]$, $\mathrm{cof}(\bigcup(a \cap \tau)) = \rho$. If G is $\mathrm{Fn}(\mu, \kappa, \mu)$-generic over M, then in $M[G]$, $\{a \in S\colon A \cap a \notin w_a\} \in NS^+_{\kappa,\lambda}$ for some $A \subseteq \mu$.*

The following is now proved as Corollary 14.4.

COROLLARY 14.6: *Assuming κ is strongly inaccessible in M, we have the following.*

(i) *If G is $\mathrm{Fn}(\kappa^+ \times \kappa, 2, \kappa)$-generic over M, then in $M[G]$, $D^*_\kappa = NS_\kappa$.*
(ii) *Assume $2^\kappa = \kappa^+$ in M, and let G be $\mathrm{Fn}(\kappa^{++} \times \kappa^+, 2, \kappa^+)$-generic over M. Then in $M[G]$, $S \notin D^*_{\kappa,\kappa^+}$ for all $S \in NS^+_{\kappa,\kappa^+}$ with*

$$S \subseteq \{a\colon \mathrm{cof}(\textstyle\bigcup a) = \mathrm{cof}(a \cap \kappa)\}.$$

Our understanding of diamond star would be much better if we could generalize the following result to uncountable cofinalities.

PROPOSITION 14.7: *In M, assume that $\lambda^{<\lambda} = \lambda$, let $w_a \in [P(a)]^{\leq |a|}$, $a \in [\lambda]^{<\kappa}$, and let $S \in NS^+_{\kappa,\lambda}$ be such that for all $a \in S$, $|a|^{\aleph_0} > |a|$ and $\mathrm{cof}(\bigcup a) = \omega$. Let G be $\mathrm{Fn}(\lambda, \lambda, \lambda)$-generic over M. Then in $M[G]$, $\{a \in S\colon A \cap a \notin w_a\} \in NS^+_{\kappa,\lambda}$ for some $A \subseteq \lambda$.*

Proof: Let $p \in G$ and F be such that p forces that $F \in \lambda^{\lambda \times \lambda}$. Let us now work in M. Let Q be the set of all $q \in \mathrm{Fn}(\lambda, \lambda, \lambda)$ such that $q \leq p$ and $\mathrm{dom}(q) \in \lambda$.

Select bijections $\varphi\colon \lambda \to Q$, $j\colon \lambda \times \lambda \to \lambda$ and $\psi\colon \lambda \to \bigcup_{\alpha\in\lambda} \lambda^{\alpha\times\alpha}$. Let D be the set of all $a \in [\lambda]^{<\kappa}$ such that

(0) $\mathrm{dom}(\varphi(\gamma)) \in a$ for all $\gamma \in a$;

(1) given α, β, γ, δ, $\xi \in a$, there are η, $\zeta \in a$ such that $\varphi(\eta) < \varphi(\gamma)$, $\varphi(\eta)(\mathrm{dom}(\varphi(\gamma))) = \delta$ and $\varphi(\eta)$ forces that $F|\xi \times \xi = \psi(\zeta)$;

(2) $j[a \times a] = a$, $a \cap \kappa \in \kappa$ and $\cup a \notin a$;

(3) $\psi(\zeta)[(a \cap \mathrm{dom}(\psi(\zeta))) \times (a \cap \mathrm{dom}(\psi(\zeta)))] \subseteq a$ for every $\zeta \in a$.

Now pick $a \in D \cap S$. Select $\chi\colon \omega \to a$ with $\bigcup_{n\in\omega} \chi(n) = \cup a$. We define $q_f \in Q$, $f \in \bigcup_{n\in\omega} a^n$, so that the following hold :

(i) $f' \subset f$ implies $q_f < q_{f'}$.

(ii) $\varphi^{-1}(q_f) \in a$.

(iii) If $f \in a^{m+1}$, then $q_f(\mathrm{dom}(q_{f|m})) = f(m)$ and q_f forces that $F[(a \cap \chi(m)) \times (a \cap \chi(m))] \subseteq a$.

Set $q_g = \bigcup_{n\in\omega} q_{g|n}$ for all $g \in a^\omega$. Now pick $g \in a^\omega$ such that $j[q_g] \cap a \notin \{b \cap j[\mathrm{dom}(q_g) \times \lambda]\colon b \in w_a\}$. Then q_g forces that $F[a \times a] \subseteq a$ and that $j[\cup G] \cap a \notin w_a$. ∎

Following again the proof of Corollary 14.4, we obtain:

COROLLARY 14.8: *Assume that in M, $\lambda^{<\lambda} = \lambda$ and κ is the successor of a cardinal of cofinality ω. Let G be $\mathrm{Fn}(\lambda^+, 2, \lambda)$-generic over M. Then in $M[G]$, $S \notin D^*_{\kappa,\lambda}$ for all $S \in NS^+_{\kappa,\lambda}$ with $S \subseteq \{a\colon \mathrm{cof}(\cup a) = \omega\}$.*

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